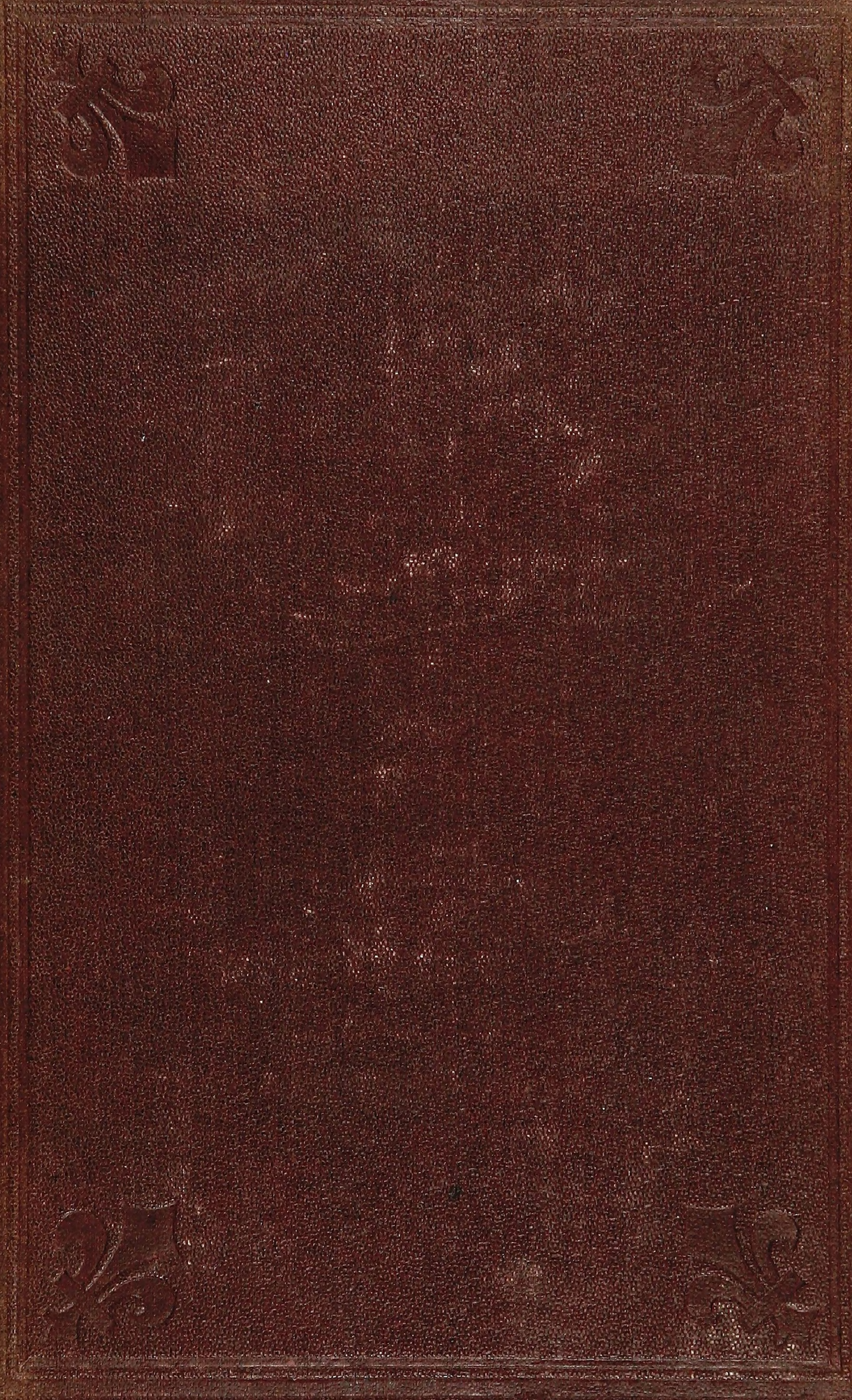

THE

ESSENTIAL ELEMENTS

OF

PRACTICAL MECHANICS.

THE

ESSENTIAL ELEMENTS
OF
PRACTICAL MECHANICS,

Based on the Principle of Work;

DESIGNED FOR

ENGINEERING STUDENTS.

BY

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PREFACE.

THIS work is designed to make good a deficiency much felt by practical engineers and engineering students engaged in civil, military, and mechanical pursuits. A student, to understand the treatises and systematic expositions, developing the theory and application of mechanics, employed as class books in our schools and colleges, requires considerable mathematical skill, and after all, the elementary knowledge acquired by studying such works is of little value to the practical engineer. On the general theory of mechanics there are many popular works clear of mathematical formulæ and serviceable to the general reader, but the practitioner cannot render them available. Again, French investigators and writers on practical mechanics take for granted that they are addressing those who understand the theory of mechanics, and when their works are copied or translated they are far from being elementary. THE ESSENTIAL ELEMENTS OF PRACTICAL MECHANICS is based upon what is termed *the principle of work*, and as work is measured by a unit, the student should be able to distinguish with ease units of work from other units. I have, therefore, introduced a simple notation and a new system to effect this object, which French writers and their copyists have neglected:—discarding the denominations *foot-pounds* and *kilogram-mètres*, which are evidently improper. The unit of work is the exact labour required to raise a pound weight through the space of one foot against the direct action of gravity. If a man takes a pound weight in his hand, and raises it one foot in the direction of the plumb-line, he will perform a unit of work.

1 lb. raised 24 feet high=**24** units of work;
2 lbs. raised 12 feet high=**24** units of work;
3 lbs. raised 8 feet high=**24** units of work;
4 lbs. raised 6 feet high=**24** units of work;
6 lbs. raised 4 feet high=**24** units of work;
8 lbs. raised 3 feet high=**24** units of work;
12 lbs. raised 2 feet high=**24** units of work;
24 lbs. raised 1 foot high=**24** units of work.

To distinguish units of work from other units, the numbers that represent units of work are printed in broad-faced type instead of the ordinary way, thus:

24 represents 24 units of work;
$\mathbf{8} \times \mathbf{3} = \mathbf{24}$;

that is the same as saying, 8 lbs. raised 3 feet high is equal to 24 units of work.

$\mathbf{24}'$; represents 24 units of work done in a minute.
$\mathbf{24}''$; represents 24 units of work done in a second.

$\mathbf{33000}' = \mathbf{1}^{h}$;

represents 33000 units of work done in a minute, equal one horse power.

Formerly, the science of mechanics was one of the last branches of applied mathematics to which a student was induced to turn his particular attention, but now the case is different. Since the wealth and power of nations depended so much on the application of the sciences to the mechanical and industrial arts, a knowledge of practical mechanics becomes of paramount importance, not only to engineering students, but also to capitalists, statesmen, and merchants, many of whom are but little skilled in the higher departments of mathematics.

While this work is designed to meet the exigencies of these changes and circumstances, and dispense with the higher geometry and an extensive use of analytical symbols, it will be found to extend the boundaries as well as simplify the practical application of this branch of philosophy, and to contain solutions of many important problems which have hitherto defied the skill of mathematicians.

This treatise, when divided as the nature of the several subjects seemed to demand, is composed of ten chapters. The *first chapter* is designed to exemplify how *work* is measured by a *unit*, both with and without reference to a unit of time. The questions selected for the purpose of illustration in this chapter, as well as those given throughout the work, are designed more to teach the student how to think, than to insist upon what he should think. However, each question may be so varied as to become a model for a variety of cases. The *second chapter* treats of the work of living agents, explains the influence of friction, and introduces one of the most beautiful laws of motion. In the *third chapter* the principles expounded in the first and second are applied to the motion of bodies on inclined planes and to the raising of materials.

The *fourth chapter* treats of the transmission of work by simple machines, establishes the properties of the lever, wheel and axle, pulley, inclined plane, wedge, and screw, on the principle of work. In this chapter the student's attention is directed to the graduation of the safety-valve lever, the centres of gravity of bodies, the hydrostatic press, and the anti-friction cam-press. In the *fifth chapter* I have given some of the most useful propositions and rules relative to the work of steam and the steam-engine; but steam being a subject in which the most elaborate theory can do but little independent of experiments, I therefore deemed it prudent, rather than delude the student with the usual display of mathematical accuracy where it is far from being attained, to give but little theoretical matter, and introduce some valuable tabulated results founded on experiments. The constant quantities are often taken in round numbers, but they are easily altered to such particular cases, without affecting the principle of work upon which the calculations are based. It may be necessary to point particular attention to the problems of this chapter, which, for the first time, are independently and accurately solved by Dual arithmetic, without the use of tables or other extraneous aids.

The *sixth chapter* is on accumulated work. The subjects stated and discussed in this chapter are: the force of gravity near the surface of the earth; units of work in a rotating body; the centre of gyration; living forces, *vis viva;* the measure of motive forces and inertia; the maximum velocity of the piston of a steam-engine; a question is solved under this head which defied the skill of mathematicians before the introduction of Dual arithmetic. In the sixth chapter I have also treated of the velocity acquired by bodies in descending an inclined plane or a curve; the crank and fly-wheel; impact; the ballistic pendulum; shock of elastic bodies; resistance of fluids; centre of oscillation, and other kindred matters.

Chapter seven is appropriated to the examination of the equilibrium of forces, pressures, tensions, and thrusts. The principle of the equality of moments, and the construction of diagrams of forces, are next discussed. The remainder of this chapter is occupied with practical examples of wooden and iron bridges, a correct theory of the strength of girders resting on supports, and with investigations respecting chain and suspension bridges, in which Dual arithmetic is again employed to obtain results not attainable by any previously known method. The comparisons drawn in this chapter with respect to different methods of construction should be calmly and closely considered. One class of mechanical investigators first establish theories on abstract reasoning, and afterwards correct them by experiments. I have taken Morin and Poncelet as representatives of this class. Another class, without any particular hypothesis, interpolate a series of experiments by empirical rules to suit particular cases within the range of such experiments: General Anstruther, alluded to in the eighth chapter, is a practical engineer of this class. A third class of investigators assume constructions which they assert possess particular properties that may or may not exist, and support such claims by high-sounding pretensions and a little misapplied mathematics to cover the delusion. The expounder of Koch's ingenious system of skeleton structures, which may or may not apply in practice, for all that this investigator has shown to the contrary, is one of this class. He assumes it possible that the structure may be composed of a series of right-angled triangles, perhaps because the calculations respecting such triangles are easily made. But in sustaining a load in motion, although these triangles are but little altered, yet it is in this little, which he disregards or explains away, lies the object of research. Such performances leave the subjects to which they are applied in doubt and abeyance. This class of investigators is very numerous, and have only succeeded in breaking down confidence in this important branch of applied mathematics.

The *eighth chapter* treats of the pressure of water and other fluids; the position of the centre of gravity in many important forms; the pressure of earth against revetment walls; and the measurement of heights by observations of the barometer and thermometer. In treating of this last subject, I have shown a simple method of solving an equation that defied the skill of mathematicians. In this chapter will be found some very important observations on the force of gravity and the constant g, which is the letter usually taken to represent it.

THE ESSENTIAL ELEMENTS OF PRACTICAL MECHANICS terminates with the *ninth chapter*. How sound theory may be made to approximate to the results of practice; the resistance of water to the motion of paddle-wheels and screw-propellers; description of the indicator; performance of paddle-wheel steamers; use of Dual logarithms; and the performance of a screw-propeller with expanding pitch, are subjects discussed in this chapter.

The student, before consulting larger works on the resistance of water to the motion of vessels, would do well to read the lectures of Captain E. G. Fishbourne on naval architecture. Here it may also be necessary to remind the reader that many productions, valuable to the engineer, are incidentally produced by liberally educated professional men who are not professed engineers; for example, the lectures of the naval officer just named on naval ordnance and iron ships saved this country from squandering large sums of the public money on guns and ships worse than useless.

The *tenth chapter* is added for the use of those who have neglected the study of common arithmetic; it contains useful arithmetical processes; operations that may be forgotten; the application of principles often misunderstood; and an introduction to Dual arithmetic. I have omitted to mention many minor topics connected with the general subjects enumerated that could not be conveniently specified in an ordinary preface; however, those topics are particularised in the Index.

OLIVER BYRNE.

PRACTICAL MECHANICS.

CHAPTER I.

ON THE UNIT OF WORK WITH, AND WITHOUT REFERENCE TO THE UNIT OF TIME.

WORK is measured by a unit, like length, weight, time, &c. To raise one pound a foot high against the earth's gravity in the direction of a plumb-line is a unit of work; it is clear, then, to raise 5 lbs. a foot high is to perform 5 units of work, and to raise 5 lbs. four feet high equal 20 units of work. Since resistance and pressure of every kind may be expressed in pounds, it follows that the unit here described may be made to measure every kind of work. It will be presently shown that a unit of work is performed whenever one pound pressure is exerted through a space of one foot, no matter in what direction the space may lie.

In developing this subject it is necessary to distinguish units of work from other units; this I propose to do by writing or setting down the numbers that represent them in black broad-faced letters and figures instead of in the ordinary way, thus,

25 represents 25 units of work.

25′ will represent 25 units of work done in a minute.

73″ will represent 73 units of work done in a second.

Sometimes the minute is taken for the unit of time, and 33 minutes is marked 33′; in other cases, the second is taken as the unit of time, and 45 seconds is marked 45″.

45″ represents 45 seconds;
but **45″** units of work done in a second of time.

a′ represents a minutes, but **a**′ represents **a** units of work done in one minute.

EXAMPLES.

Question 1. Required the units of work expended in raising a weight of 50 lb. to the height of 31 ft.

Units of work in raising 1 lb. 31 feet=**31**.

∴ 50×**31**=**1550**,

the required units of work.

Ques. 2. The ram of a pile-driving engine weighs half a ton, and has a fall of 17 feet, how many units of work are performed in raising this ram?

Half a ton=1120 lbs.

Units of work in raising 1 lb. 17 feet=**17**.

∴ 1120×**17**=**19040**,

the units of work required to raise 1120 lbs. to a height of 17 feet.

Ques. 3. How many units of work are required to raise 7 cwt. of coal from a pit whose depth=13 fathoms?

7×112=784 lbs.

6 feet being the length of a fathom,

13×6=78 feet, in 13 fathoms.

Hence the work consists in raising 784 lbs. to the height of 78 feet.

∴ 784×78=**61152**.

Ques. 4. If the weight of a man be 183 lbs., and if he ascends a perpendicular height of 20 feet, he does work in raising himself, what are the number of units?

In this operation the man raises the weight of his own body,

∴ 183×20=**3660**,

amount of work.

If this man were to descend in a bucket, it is clear he would perform the same work upon a counterpoise weight when he has descended 20 feet.

Ques. 5. How many units of work will be required to pump 8000 cubic feet of water from a mine whose depth = 500 fathoms?

A cubic foot of water weighs 62·5 lbs.

$$8000 \times 62{\cdot}5 = 500000 \text{ lbs.}$$

$$500 \text{ fathoms} = 3000 \text{ feet.}$$

Consequently the work is to raise 500000 lbs. a perpendicular height of 3000 feet.

$$\text{Work} = 500000 \times 3000 = \mathbf{150000000}.$$

Ques. 6. A horse draws 150 lbs. out of a well, by means of a rope going over a fixed pulley, moving at a rate of 2½ miles an hour, how many units of work does this horse perform a minute, the friction being neglected?

A mile = 5280 feet, then,

$$\frac{2\frac{1}{2} \times 5280}{60} = 220 \text{ feet passed over per minute.}$$

$$\therefore \quad \text{Work per minute} = 150 \times 220 = \mathbf{33000'}.$$

$$\text{Work per second} = \mathbf{550''}.$$

THE UNIT OF WORK REFERRED TO A UNIT OF TIME.

A unit of work, or **1**, represents 1 lb. raised 1 foot.

A unit of work in a minute, or **1′**, represents 1 lb. raised a foot high in a minute.

A unit of work in a second, or **1″**, represents 1 lb. raised a foot high in a second.

It has been assumed that a horse is capable of raising 33000 lbs. a foot high in a minute, or to perform **33000** units of work in a minute.

Hence a horse power = $\mathbf{33000' = 1^h}$, or **1** HP.

28 horse power decimal 37 is written $\mathbf{28^h{\cdot}37}$, or **28·37** HP.

Whether this power is greater or less than the power of a horse it matters little, while it is a power so well defined.

To raise 10000 lbs. a foot high in a minute, would be a more convenient unit to measure by.

EXAMPLES.

Question 1. How many horse power would it take to raise 3 cwt. of coal a minute from a pit whose depth = 110 fathoms?

$$\text{Depth} = 110 \times 6 = 660 \text{ feet.}$$

$$3 \text{ cwt.} = 112 \times 3 = 336 \text{ lbs.}$$

$$660 \times 336 = \mathbf{221760}'.$$

Since a horse power = $\mathbf{33000}'$

$$\frac{\mathbf{221760}'}{\mathbf{33000}'} = \mathbf{6 \cdot 72} \text{ HP., the required power.}$$

6·72 HP. stands for 6 horse power decimal 72; the student should observe that the full point which separates whole numbers from decimals is placed above (**·72**), but when otherwise employed it is put below (HP.)

·72 = the vulgar fraction $\frac{72}{100}$, a cipher being put under each decimal figure and 1 under the full point.

Ques. 2. How many horse power is required to raise 2200 cubic feet of water an hour from a mine whose depth = 63 fathoms?

$$137500 = 2200 \times 62 \cdot 5 = \text{weight of water in lbs.}$$

$$63 \times 6 = 378 \text{ feet, depth of mine.}$$

$$\frac{137500 \times 378}{60} = \mathbf{866250}'.$$

$$\therefore \quad \frac{866250}{330\cancel{00}} = \mathbf{26}\tfrac{1}{4} \text{ HP.}$$

The proposed notation must be borne in mind, $\mathbf{866250}'$ signifies 866250 lbs., raised one foot high in a minute, and $\mathbf{26\frac{1}{4}}^{h}$, represents $26\frac{1}{4}$ horse power, or $26\frac{1}{4}$ times 33000 lbs. raised 1 foot high in a minute.

Ques. 3. A winding engine is moved by 4 horses, what weight of coal will be raised an hour from a pit whose depth = 200 feet?

Work of the 4 horses in an hour,

$$= 4 \times 33000 \times 60 = \mathbf{7920000}.$$

Work in raising 1 lb. of coal 200 feet = **200**.

Since it takes 200 units of work to raise a pound of coals,

$$\frac{7920000}{200} = 39600 \text{ lbs.}$$

Consequently, four horses will raise 39600 lbs. of coal, or of anything else, a height of 200 feet in an hour.

Ques. 4. In what time will an engine of 10-horse power raise 5 tons of material from the depth of 132 feet?

$$5 \text{ tons} = 11200 \text{ lbs.}$$

Work of the engine per minute,

$$= \mathbf{330000}' = 33000 \times 10.$$

Work of raising 5 tons,

$$= 11200 \times 132 = \mathbf{1478400}.$$

Because the engine performs 330000 units of work a minute,

$$\therefore \quad \frac{1478400}{330000} = 4\cdot48 \text{ minutes.}$$

Ques. 5. How many cubic feet of water will an engine of 36-horse power ($\mathbf{36}^{h}$) raise in an hour from a mine whose depth is 40 fathoms?

$$40 \times 6 = 240 \text{ feet.}$$

Work in an hour,

$$= 36 \times 33000 \times 60 = \mathbf{71280000}.$$

Work to raise a cubic foot of water 240 feet,

$$= 240 \times 62\cdot5 = \mathbf{15000}.$$

$$\frac{\mathbf{71280000}}{\mathbf{15000}} = 4752 \text{ cubic feet.}$$

Ques. 6. From what depth will an engine of 22-horse power raise 13 tons of coal in an hour?

Work done by the engine in an hour,

$$22 \times 33000 = \mathbf{726000}.$$

$$13 \text{ tons} = 29120 \text{ lbs.}$$

$$\frac{\mathbf{726000}}{29120} = 24\cdot9 \text{ feet.}$$

Ques. 7. An engine is observed to raise 7 tons of material an hour, from a mine whose depth is 85 fathoms; required the horse power of the engine, supposing $\frac{1}{6}$ of its work to be lost in transmission?

$$\text{Work a min.} = \frac{2240 \times 7 \times 85 \times 6}{60} = \mathbf{133280}'.$$

Since $\frac{5}{6}$ of the work of the engine only go to raise the material,

$$\therefore\quad \tfrac{5}{6} \text{ of } \mathbf{33000'=27500'},$$

the units of useful work of one horse power a minute.

$$\frac{\mathbf{133280'}}{\mathbf{27500'}}=\mathbf{4^{h}\cdot4829}.$$

Ques. 8. Required the horse power of an engine that would supply the city of Brooklyn with water, working 12 hours a day, the water to be raised to a height of 50 feet; the number of inhabitants=130000, and each person to use 5 gallons of water a day?

The standard gallon of the United States weighs $8\frac{1}{3}$ lbs., nearly.

$$\frac{130000\times5=8\frac{1}{3}}{12\times60}=\frac{8125\times25 \text{ lbs.}}{27}$$

the pounds of water to be pumped in a minute.

$$\therefore\quad \frac{8125\times25}{27}\times50=\mathbf{376157'}$$

$$\therefore\quad \frac{\mathbf{376157'}}{\mathbf{33000'}}=\mathbf{11^{h}\cdot4}.$$

The horse power required?

Ques. 9. What is the horse power of an engine that pumps from three different levels, whose depths are 40, 70, and 90 fathoms respectively; from the first, 20 cubic feet of water are raised a minute; from the second, 10 cubic feet; and from the third, 35 feet, allowing $\frac{1}{3}$ the work of the engine to be destroyed by useless resistance?

$$\mathbf{1^{h}}\quad=\mathbf{33000'}.$$

$$\tfrac{2}{3} \text{ of } \mathbf{33000'=22000'},$$

the effective power of the engine.

$$\begin{array}{lllr}
\text{1st level, work} & & =62\cdot5\times20\times240= & \mathbf{300000'}. \\
\text{2nd} \quad ,, & ,, & =62\cdot5\times10\times420= & \mathbf{262500'}. \\
\text{3rd} \quad ,, & ,, & =62\cdot5\times35\times540= & \underline{\mathbf{1181250'}}. \\
 & & & \mathbf{1743750'}.
\end{array}$$

$$\therefore\quad \frac{\mathbf{1743750}}{\mathbf{22000'}}=\mathbf{79^{h}\cdot26}.$$

Ques. 10. There are 6000 cubic feet of water in a mine whose depth is 60 fathoms, when an engine of 50 horse power began to work the pump; the engine continued to work 5 hours before the mine was cleared of the water; required the number of cubic feet of water which had run into the mine per hour, supposing $\frac{1}{4}$ of the work of the engine to be lost by transmission?

4)**33000′**
8250′
24750′ effective power of the

engine a minute. It must not be forgotten that **24750′** written in broad-faced figures, signifies 24750 units of work in a minute or 24750 lbs., raised one foot high in a minute; so that setting down the numbers in this peculiar manner saves much written explanation.

$$\mathbf{24750'}\times 50\times 5\times 60=\mathbf{371250000},$$

effective work of the engine in 5 hours.

$$62{\cdot}5\times 60\times 6=\mathbf{2250},$$

work in raising one cubic foot of water to a height of 60 fathoms =360 feet.

$$\frac{\mathbf{371250000'}}{\mathbf{22500}}=16500{\cdot}$$

cubic feet of water pumped in 5 hours.

16500·
6000·
5) 10500· Water run in during 5 hours.
21000·

Cubic feet run in one hour.

Ques. 11. A forge hammer weighs 300 lbs., makes 100 lifts a minute, the perpendicular height of each lift=2 feet; what is the horse power of the engine that gives power to 20 such hammers?

Work of each lift=300 × 2 × 20=**12000**.

Work in 100 lifts, that is, in one minute,

$$\mathbf{12000}\times 100=\mathbf{1200000'}.$$

$$\frac{\mathbf{1200000'}}{\mathbf{33000'}}=\mathbf{36^{h}\cdot 36}.$$

Ques. 12. An engine of 10 horse power (**10^h**), raises 4000 lbs. of coal from a pit 1200 feet deep in an hour, and also gives motion to a hammer which makes fifty lifts in a minute, each lift having a perpendicular height of 4 feet, what is the weight of the hammer?

Work done by the engine in one minute,

$$=\mathbf{33000}\times 10=\mathbf{330000'}.$$

The units of work performed, in raising the coals, a minute=

$$\frac{1200\times 4000}{60}=\mathbf{80000'}.$$

Work engaged in raising the hammer a minute,

$$=\mathbf{330000'}-\mathbf{80000'}=\mathbf{250000'}.$$

Work a minute, in raising one lb. of hammer 4 feet high, 50 times a minute,=

$$1\times 4\times 50=\mathbf{200'}.$$

$$\frac{\mathbf{250000'}}{\mathbf{200'}}=1250 \text{ lbs.},$$

the weight of the hammer required.

CHAPTER II.

ON THE WORK OF LIVING AGENTS.

THE labouring force of animals varies very much with the way in which their muscular strength is exerted; and also, with the rate at which they labour.

The following little table shows the greatest amount of effective work that a labouring man can perform under the different modes in which he may exert his muscular power.

WORK DONE BY A MAN PER MINUTE, WHEN HE WORKS **8** HOURS A DAY.

A man in raising his own body	**4250'**.
A man in working at a treadmill	**3900'**.

A man drawing or pushing horizontally	**3120′.**
A man pushing or pulling vertically	**2380′.**
A man turning a handle	**2600′.**
A man working with arms and legs, as rowing .	**4000′.**

UNITS OF WORK DONE BY A MAN PER MINUTE, WHEN HE WORKS 6 HOURS A DAY.

A man in raising material with a pulley . . .	**1560′.**
A man in raising materials with the hands . . .	**1470′.**
A man in raising material on back, returning empty	**1126′.**

UNITS OF WORK DONE BY A MAN PER MINUTE, LABOURING 10 HOURS A DAY.

Raising materials with a wheelbarrow on ramps .	**720′.**
Throwing earth to the height of 5 feet	**470′.**

UNITS OF USEFUL WORK DONE BY A MAN IN A MINUTE, RAISING WATER BY DIFFERENT ENGINES, WORKING 8 HOURS A DAY.

With a windlass from deep wells	**2560′.**
With an upright chain pump	**1730′.**
With a treadmill	**3176′.**
With a Chinese wheel	**2167′.**
With an Archimedian screw	**1505′.**
Raising water from a well, with rope and pail . .	**1054′.**

WORK OF BEASTS.

THE imaginary horse by which the power of steam-engines is measured, is supposed to do more work than the general run of horses are able to perform. The work of a horse of average strength is about 22000 pounds raised one foot high in a mintue, which according to the author's notation, is expressed thus:—

22000′.

A mule will perform $\frac{2}{3}$ the work of a horse=**14666$\frac{2}{3}$′.**

An ass will perform about $\frac{1}{5}$ the work of a horse=**4400′**.

In a common pumping engine a horse of average strength will do only 17550 useful units of work in a minute.

$$\begin{array}{rr} \text{From} & \mathbf{22000'} \\ \text{Take} & \mathbf{17550'} \\ \hline & \mathbf{4450'} \end{array}$$

4450′, lost in this case by friction and useless resistances.

EXAMPLES.

Question 1. How many cubic feet of clay weighing 100 lbs. the cubic foot, will 20 men throw 5 feet high in a day of 10 hours long?

From the foregoing table the number **470′** is found.

Work in a day,

$$=\mathbf{470'}\times 60\times 10\times 20=\mathbf{5640000'}.$$

Work in raising one cubic foot,

$$=100\times 5=\mathbf{500'}.$$

Consequently the number of cubic feet raised in one day by 20 men,

$$=\frac{\mathbf{5640000}}{\mathbf{500}}=11280$$

Ques. 2. How many bricks will a man raise in a day six hours long to a height of 30 feet, supposing the weight of a cubic foot=125 lbs., and 17 bricks to form a cubic foot?

For a man working in this way, the foregoing tables give **1126′**.

$$\therefore \quad \mathbf{1126'}\times 60\times 6=\mathbf{405360},$$

work done by a man in 6 hours.

$$125\times 30=\mathbf{3750},$$

work done in raising a cubic foot 30 feet high.

$$\therefore \quad \frac{\mathbf{405360}}{\mathbf{3750}}=108{\cdot}096,$$

cubic feet raised in a day of 6 hours.

Number of bricks=108·096 × 17=1838.

Ques. 3. How many cubic feet of water will a labourer, working with a bucket and rope, raise from a well whose depth is 16 feet?

In the table will be found **1054′**, for a man working in this manner.

$$\mathbf{1054'} \times 60 \times 8 = \mathbf{505920},$$

work done in 8 hours.

The work done in raising a cubic foot of water 16 feet high,

$$= 6{\cdot}25 \times 16 = \mathbf{1000}.$$

Hence the cubic feet raised in 8 hours by a man,

$$= \frac{\mathbf{505920}}{\mathbf{1000}} = 505{\cdot}92.$$

Ques. 4. What weight is a man, working at a treadmill, able to raise in a day of 8 hours, from a depth of 110 feet?

$$\text{Tabular number} = \mathbf{1730'}.$$

$$\mathbf{1730'} \times 60 \times 8 = \mathbf{830400},$$

work done in a day.

The work required to raise one cwt. or 112 lbs. a height of 110 feet will be,

$$112 \times 110 = \mathbf{12320};$$

$$\therefore \quad \frac{\mathbf{830400}}{\mathbf{12320}} = 67{\cdot}4 \text{ cwt.},$$

the weight that may be raised by a man working at a treadmill.

Ques. 5. How many tons of coal would a man raise, working with a wheel and axle, from a pit whose depth is 20 feet, taking for granted, according to the tabulated statement, that a man so circumstanced can perform 2600 units of work in a minute?

The work done in one day of 8 hours,

$$= \mathbf{2600'} \times 60 \times 8 = \mathbf{1248000}.$$

The work to raise one ton,

$$= 2240 \times 20 = \mathbf{44800};$$

$$\therefore \quad \frac{\mathbf{1248000}}{\mathbf{44800}} = 27\tfrac{6}{7} \text{ tons}$$

raised in a day of 8 hours.

Ques. 6. The ram of a pile engine weighs 7 cwt., and has a fall of 23 feet, how many strokes a day will four men give, working a wheel and axle?

Work done in one day of 8 hours,

$$=\mathbf{2600}'\times 60\times 8\times 4=\mathbf{4992000}.$$

The work of one stroke,

$$=112\times 7\times 23=\mathbf{18032};$$

$$\therefore\ \frac{\mathbf{4992000}}{\mathbf{10832}}=277 \text{ strokes nearly.}$$

The force with which animals pull decreases with their speed; the relation between the traction and speed of a horse is expressed with considerable accuracy by the following formula:

$$t=250-41\tfrac{2}{3}\,r;$$

in which t=the traction in lbs. and r=the rate in miles per hour.

Hence, if the speed be 3 miles an hour, the traction will be 125 lbs., for

$$250-41\tfrac{2}{3}\times 3=125 \text{ lbs.}$$

If the speed be one mile an hour, the traction will be=208⅓ lbs. for

$$250-41\tfrac{2}{3}\times 1=208\tfrac{1}{3} \text{ lbs.}$$

From this formula, it is easily shown that a horse will do the greatest amount of work when he travels at the rate of 3 miles an hour, in fact

$$\left\{250-41\tfrac{2}{3}\,r\right\}\times r$$

becomes a maximum when $r=3$.

If a body move slowly along a horizontal plane, the resistance to be overcome is called friction. Experiment has determined that this resistance on a given surface is a fractional part of the weight of the body moved, and it has also been found that any change in the rate of the motion of the body does not affect the resistance due to friction; nor is the amount of friction altered by the extent of the rubbing surfaces.

When a waggon or a cart is drawn along a good common road, the resistance of friction is about $\frac{1}{30}$ of the whole load, so that a horse in order to draw 3120 lbs. along a road, must pull with the force of 104 lbs., for

$$\frac{3120}{30}=104 \text{ lbs.}$$

A horse with this traction, according to the foregoing formula, moves at the rate of about $3\frac{1}{2}$ miles an hour, for

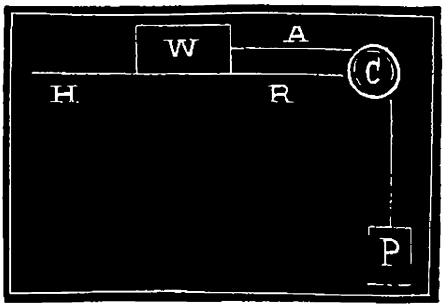
$$t = 250 - 41\tfrac{2}{3}\,r,$$

$$\text{becomes } 104 = 250 - 41\tfrac{2}{3}\,r,$$

$$\therefore\ r = 3\tfrac{1}{2}.$$

A carriage on a railroad only requires a pressure of about $\frac{1}{300}$ part of the moving weight to give it motion, or from 4 to 8 lbs. a ton. The fractions $\frac{1}{30}$ for common roads, and $\frac{1}{300}$ for railroads, are called the coefficients of friction; as these coefficients become smaller, the rubbing surfaces become smoother.

Fig. 1.



Let W be a weight drawn on a horizontal plane, H R, by means of a weight, P, attached to a cord, A, going over a fixed pulley, C; then the weight, P, just necessary to draw or move W along the plane, will be equal to the resistance of friction. In the case of a railroad, if W = 150 tons, P will be = 900 lbs., when the coefficient of friction = $\frac{3}{1120}$, or 6 lbs. a ton.

It is very evident that whatever distance P descends, the weight, W, will be drawn along the plane, H R, the same distance; hence the units of work done in moving W will be the weight of P in pounds multiplied by the distance in feet through which it descends, or the resistance of friction in pounds, multiplied by the space in feet over which W is moved.

The work of every machine is consumed by the work done, or by the useful work, together with the useless work, or the work destroyed by the friction of the parts of the machine. I will here explain one of the most beautiful laws of motion: When the work applied exceeds the work consumed, the redundant work goes to increase the speed of the parts of the machine, and at the same time, like the fly-wheel, acts as a reservoir of work. *This acceleration goes on increasing until the work of the resistances + the useful work = the work applied; and then the motion of the machine becomes uniform.*

For example, in a railroad engine and train, at first the work of the engine exceeds the work of the resistances, and hence the speed of the engine goes on increasing; but, as the speed increases, the work of the resistances also increases, so that ultimately the engine attains a nearly uniform motion, which is called the greatest or maximum speed, and then the work destroyed by the resistances will be exactly equal to the work applied by the moving power.

Ques. 7. Required the effective horse power of a locomotive engine which moves at a steady speed of 23 miles an hour upon a level rail, the weight of the train being 100 tons, and the friction 5 lbs. a ton?

Put x=the required horse power.

The work of the engine per minute,

$$=x\times 33000';$$

The resistance of friction,

$$=5\times 100=500 \text{ lbs.}$$

The distance moved per minute,

$$=\frac{23\times 5280}{60}=2024 \text{ feet};$$

Work to overcome the friction,

$$=2024\times 500=1012000'.$$

But as the speed of the train is uniform, the work of the resistances will be equal to the effective work of the engine;

$$\therefore\quad x\times 33000'=1012000'.$$

$$\therefore\quad x=\frac{1012000}{33000'}=30^{h}\cdot 7.$$

Ques. 8. What is the rate in miles per hour of a train of 80 tons, drawn by an engine of 70^{h}; the friction 8 lbs. to the ton?

Call x the uniform speed in miles per hour;

Work used in moving the train x miles,

$$=80\times 8\times 5280\times x;$$ this is the work done by the engine in an hour. But the work done by the engine in an hour will also be expressed by

$$33000'\times 70\times 60;$$

$$\therefore\quad 33000'\times 70\times 60=80\times 8\times 5280\times x;$$

$$\therefore\quad x=\frac{33000'\times 70\times 60}{80\times 8\times 5280}=41\cdot 02 \text{ miles.}$$

Ques. 9. An engine of 48^{h} moves with a maximum speed of 33 miles an hour, on a level rail; required the gross load of the train, friction 6 lbs. a ton?

Let x be the gross weight of the train in tons, then the work consumed an hour in moving the train.

$$=x\times\mathbf{6}\times\mathbf{33}\times\mathbf{5280}.$$

Work of the engine an hour,

$$=\mathbf{48}\times\mathbf{33000'}\times\mathbf{60}.$$

When the speed is uniform or at its maximum,

$$x\times\mathbf{6}\times\mathbf{33}\times\mathbf{5280}=\mathbf{48}\times\mathbf{33000}\times\mathbf{60};$$

$$\therefore\quad x=\frac{48\times 33000\times 60}{6\times 30\times 5280}=90\tfrac{10}{11}\text{ tons.}$$

Ques. 10. In what time will an engine of 66 horse power, moving a train of 200 tons, complete a journey of 100 miles, friction 5 lbs. per ton; rails horizontal?

Work expanded in moving the train 100 miles,

$$=100\times 5280\times 200\times 5=\mathbf{528000000}.$$

Work of the engine per hour,

$$=\mathbf{33000'}\times 66\times 60=\mathbf{130680000}$$

$$\frac{\mathbf{528000000}}{\mathbf{130680000}}=4{\cdot}04\text{ hours.}$$

Ques. 11. What work per minute will a horse perform when travelling at the rate of $2\frac{1}{2}$ miles an hour?

I have before shown that the traction of a horse moving at the rate of $2\frac{1}{2}$ miles an hour,

$$=250-41\tfrac{2}{3}\times 2\tfrac{1}{2}=145\tfrac{5}{6}\text{ lbs.}$$

$$2\tfrac{1}{2}\text{ miles}=13200;$$

$$\text{Feet per minute}=\frac{13200}{60}=220\text{ feet.}$$

Hence the work of this horse per minute,

$$=145\tfrac{5}{6}\times 220=\mathbf{32083\tfrac{1}{3}'}.$$

Ques. 12. What load will a horse draw travelling at 3 miles an hour upon a plank road, whose friction is $\frac{1}{100}$ of the whole load, the road being horizontal?

The traction at this speed is 125 lbs.,

$$\text{for } 250 - 41\tfrac{2}{3} \times 3 = 125.$$

The gross load must be 100 times this weight,

$$= 125 \times 100 = 12500 \text{ lbs.}$$

Ques. 13. Suppose a horse to be able to perform 33000 units of work in a minute on a horizontal road, whose friction is $\frac{1}{20}$ of the whole road; required the load and rate per hour?

$$\text{Traction} = (250 - 41\tfrac{2}{3}\, r) \text{ pounds.}$$

Space passed over in a minute,

$$= \frac{5280\, r}{60} = 88\, r \text{ feet.}$$

Then $(250 - 11\frac{2}{3}\, r)\, 88\, r =$ the units of work performed by the horse in a minute $=$ **33000**′. From this quadratic equation, the value of r is found to be 3 miles;

$$\therefore \quad \text{Traction} = 125 \text{ lbs.,}$$

and the load will be

$$125 \times 20 = 2500 \text{ lbs.}$$

Ques. 14. At what rate will a horse draw a ton, on a road whose coefficient of friction is $\frac{1}{32}$?

$$\text{Traction} = \frac{2240}{32} = 70 \text{ lbs.}$$

$$\therefore \quad 70 = 250 - (41\tfrac{2}{3})\, r;$$

$$\therefore \quad r = \frac{250 - 70}{41\frac{2}{3}} = 4{\cdot}32,$$

the rate is 4·32 miles an hour.

Ques. 15. When a horse exerts a traction of $41\frac{2}{3}$ lbs., his rate of motion is 5 miles an hour, what gross load will he draw on a level road whose coefficient of friction is $\frac{1}{30}$, and what work will he perform a minute?

$$\text{Load} = 30 \times 41\tfrac{2}{3} = 1250 \text{ lbs.}$$

Distance moved in feet per minute,

$$= \frac{5 \times 5280}{60} = 440 \text{ feet.}$$

Work $= 41\frac{2}{3} \times 440 =$ **18333⅓**′.

Ques. 16. What must be the horse power of an engine, to cut 6600 square feet of planking in a day of 10 hours long?

It requires, according to experiment, about 29000 units of work to saw a square foot of green oak planking, therefore,

The work in cutting 6600 square feet,

$$=29000\times 6600=191400000.$$

$$\therefore\ \frac{191400000}{10\times 60}=319000',$$

$$\text{and}\ \frac{319000'}{33000'}=9\tfrac{2}{3}\ \text{HP}.$$

Ques. 17. An engine of 24 effective horse power cuts 144 square feet of American live oak in 5 minutes, how many units of work are consumed in cutting a square foot?

The work of the engine in five minutes,

$$=33000'\times 5\times 24=3960000,$$

the work destroyed in cutting 144 square feet.

Therefore the units of work expended in cutting one square foot,

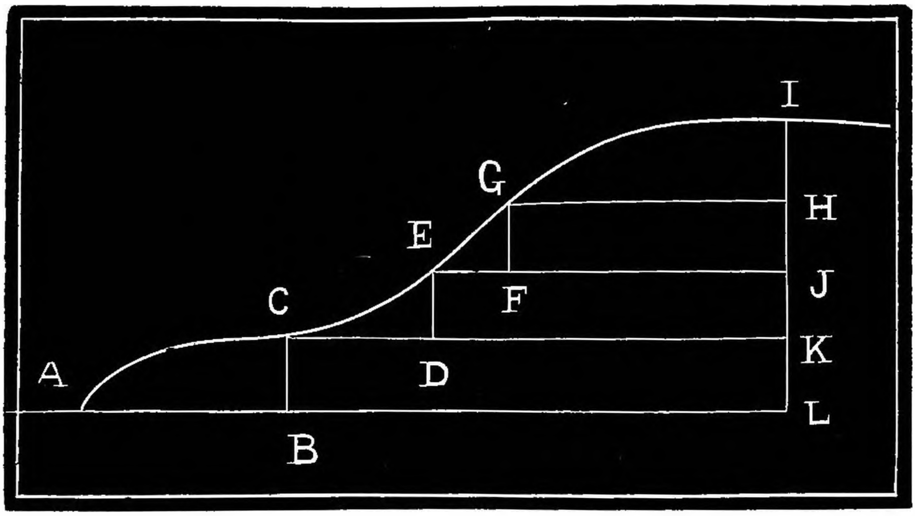
$$=\frac{3960000}{144}=27500.$$

CHAPTER III.

ON THE MOVING OF BODIES ON INCLINED PLANES, AND THE RAISING OF MATERIALS.

If a surface be supposed without friction, the units of work performed by moving a body along it is equal to the product of the weight of the body in pounds, by the vertical height in feet through which it is raised.

Fig. 2.



To move a body along the path A C E G I, may be imagined the same as to move it along and up an infinite number of steps, resembling A B, B C, C D, D E, &c. And since friction is neglected, there is no work required to move the body in a horizontal direction, all the work is expended in raising the body in the perpendicular direction; hence the units of work required to raise a body from L to I equals the units of work required to move it along the path A C E G I, the friction of the path being neglected.

The principle here explained holds true for inclined planes.

EXAMPLES.

Question 1. A train of 200 tons ascends an incline which has a rise of 5 feet in 1000, with a uniform speed of 30 miles an hour, what is the effective power of the engine, the friction being $5\frac{1}{2}$ lbs. to the ton?

The pressure of a body on an inclined plane is nearly equal to its weight, when the inclination is small, hence the work due to friction may be found by the method explained in the last chapter.

Speed of train a minute $= 2640$ feet.

Weight of train in lbs. $= 448000$.

$$\frac{5}{1000} = \frac{1}{200}, \text{ rise}$$

of the rail in one foot.

$$\frac{1}{200}\times 2640 = 13{\cdot}2 \text{ feet},$$

the rise of the rail in 2640 feet,

Consequently the whole weight of the train is raised 13·2 feet every minute in opposition to gravity.

Work due to gravity a minute,

$$= 448000 \times 13{\cdot}2 = \mathbf{5913600'}.$$

Work due to friction a minute,

$$= 200 \times 5\tfrac{1}{2} \times 2640 = \mathbf{2904000'}.$$

Total work of the engine a minute,

$$= \mathbf{5913600 + 2904000 = 8817600'}.$$

$$\frac{\mathbf{8817600'}}{\mathbf{33000}} = \mathbf{267^{h}\cdot 2}.$$

Ques. 2. A train of 330 tons ascends an incline that has a rise of $\frac{1}{5}$ in 100, what is the maximum speed with an engine of 120 horse power; the friction of the rail 8 lbs. per ton?

Let x be the speed of the train in feet per hour.

The rise in each foot of rail,

$$= \tfrac{1}{5} \text{ divided by } 100 = \tfrac{1}{500};$$

the rise of the rail in x feet,

$$= \frac{x}{500};$$

The work due to gravity in an hour,

$$= 330 \times 2240 \times \frac{x}{500} = \mathbf{1478{\cdot}4} \times x;$$

Work due to friction in an hour,

$$330 \times 8 \times x = \mathbf{2640} \times x.$$

It is evident that the work due to gravity in an hour, added to the work due to friction in an hour, must be equal to the work done by the engine in an hour,

$$\therefore \quad \mathbf{33000'} \times 120 \times 60 = \mathbf{2640} \times x + \mathbf{1478{\cdot}4} \times x,$$

$$\therefore \quad x = \frac{\mathbf{237600000}}{\mathbf{4118{\cdot}4}} = 57692 \text{ feet}.$$

Ques. 3. An engine of 50 horse power ascends a gradient having a rise of $\frac{3}{4}$ in 100; with a steady speed of 20 miles an hour, what is the weight of the train in tons, the friction per ton being 8 lbs.?

Let x be the weight of the train in tons; the work required to overcome friction,

$$=1760\times 8\times x=\mathbf{14080}\times x\,.$$

Since a speed of 20 miles an hour $=1760$ feet a minute

rise of rail in a foot,

$$=\tfrac{3}{4}\text{ divided by }100=\tfrac{3}{400};$$

rise of the rail in 1760 feet,

$$=\frac{5280}{400}=13{\cdot}2\text{ feet};$$

work due to gravity,

$$=2240\times x\times 13{\cdot}2=\mathbf{29568}\times x'$$

work of the engine a minute,

$$=\mathbf{33000'}\times 50=\mathbf{165000'},$$

Consequently,

$$\mathbf{14080}\,x'+\mathbf{29568}\,x'=\mathbf{1650000'}$$

$$\therefore\quad x=\frac{\mathbf{1650000}}{\mathbf{43648}}=37{\cdot}8\text{ tons.}$$

Ques. 4. A train of 100 tons *descends* a gradient having a rise of a $\frac{1}{4}$ of a foot in 100 feet, at a uniform speed of 60 miles an hour, what is the horse power of the engine, friction reckoned at 8 lbs. a ton?

60 miles an hour $=5280$ feet a minute;

Work due to friction,

$$=100\times 5280\times 8=\mathbf{4224000'}.$$

Rise in one foot,

$$=\tfrac{1}{4}\text{ divided by }100=\tfrac{1}{400}.$$

Rise in 5280 feet, or in one minute,

$$=\frac{5280}{400}=13{\cdot}2\text{ feet.}$$

Work due to gravity,

$$=224000\times 13{\cdot}2=\mathbf{2956800'}.$$

In this case gravity acts with the engine;

$$\therefore\quad \mathbf{4224000'}\text{ minus }\mathbf{2956800'}=\mathbf{1267200'}$$

$$\frac{\mathbf{1\ 67200'}}{\mathbf{33000'}}=\mathbf{38}^{\text{h}}\cdot\mathbf{4}.$$

Ques. 5. If a horse exerts a traction of 144 lbs., what weight can he pull on a plank road, up a hill that has a rise of 3 feet in 190 feet, supposing the coefficient of friction to be $\frac{1}{20}$?

Work of the horse in moving over 190 feet,

$$=190\times144=\mathbf{27360}.$$

The work of friction in moving x lbs. over 190 feet,

$$\frac{x}{20}\times190=\mathbf{9\cdot5}\,x,$$

supposing x to be the required load;

The work due to gravity, when the load is moved over 190 feet $=\mathbf{3}\,x$;

$$\therefore\quad \mathbf{9\cdot5}\,x+\mathbf{3}\,x=\mathbf{27360};$$

$$\therefore\quad x=\frac{\mathbf{27360}}{\mathbf{12\cdot5}}=2188\cdot8\text{ lbs.}$$

Ques. 6. What would be the backward pressure of a horse in going down hill that has a rise of 15 feet in 369 with a load of 2000 lbs., supposing the coefficient of friction to be $\frac{1}{30}$?

Work due to gravity in moving 2000 lbs. 369 feet,

$$=2000\times15=\mathbf{30000};$$

Work due to friction in moving the load 369 feet,

$$=\frac{2000}{30}\times3\cdot69=\mathbf{24600};$$

From **30000**
Take **24600**
5400

$$\therefore\quad \frac{\mathbf{5400}}{369}=14\cdot6\text{ lbs.}$$

Ques. 7. How many horses would it take to draw a load of 6 tons up a hill having a rise of $2\frac{1}{2}$ in 100, supposing the resistance of friction to be $\frac{1}{12}$ of the whole load, and the traction of each horse 160 lbs.

Let x be the number of horses.

Work due to friction in moving 6 tons over 100 feet,

$$=\frac{6\times 2240}{12}\times 100=\mathbf{112000}.$$

Work of gravity in moving 6 tons over 100 feet,

$$=6\times 2240\times 2\tfrac{1}{2}=\mathbf{33600}.$$

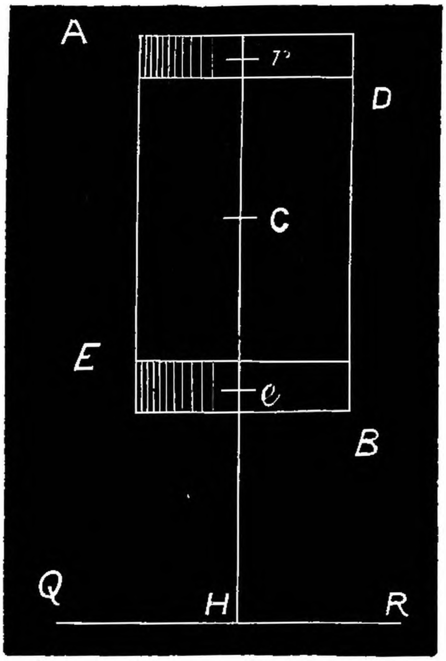
The work of x horses in passing over 100 feet of this road,

$$=160\times x\times 100=\mathbf{16000}\,x;$$

$$\therefore\quad \mathbf{16000}\,x=\mathbf{112000}+\mathbf{83600},$$

$$\therefore\quad x=\frac{\mathbf{145600}}{\mathbf{16000}}=9{\cdot}1 \text{ horses}.$$

Fig. 3.



The work in raising materials, having a given form, will be their weight multiplied by the perpendicular height to which the centre of gravity is raised.

Suppose the body, A B, to be raised from the horizontal line, Q R, and the point, C, to be the middle or centre of gravity of the body; let e and r be points equidistant from C, and the centres of gravity of two equal solices A D, E B. Now if equal weights, say for example 5 lbs., be placed at e and r, the centre of gravity of these weights will be at C, then

Work in raising 5 lbs. to $r=5\times \mathrm{H}\,r$.

" " " to $e=5\times \mathrm{H}\,e$.

Sum $\underline{=5\times(\mathrm{H}\,r+\mathrm{H}\,e)}$.

But $\mathrm{H}\,r+\mathrm{H}\,e=2\ \mathrm{H\,C}$;

$\therefore$ sum$=10\times \mathrm{H\,C}$.

That is, 10 lbs. raised from H to C is the same amount of work as 5 lbs. raised to *r*, and 5 lbs. raised to *e*. The same may be proved of any two equal weights, one of them raised as far above the centre of gravity C, as the other is below it.

Ques. 8. Required the units of work in raising the material of a wall 22 feet long, 13 feet high, and $2\frac{1}{2}$ feet thick; supposing the weight of a cubic foot of the material to be 140 lbs.?

Contents of the wall in cubic feet,

$$=22\times 13\times 2\tfrac{1}{2}=715.$$

Weight of the wall in pounds,

$$=715\times 240=10010.$$

The height of the centre of gravity of the whole wall,

$$=\frac{13}{2}=6\tfrac{1}{2}\text{ feet};$$

$$\text{Work}=10010\times 6\tfrac{1}{2}=\mathbf{65065}.$$

Ques. 9. The shaft of a pit is to be sunk 120 feet deep, and 6 feet in diameter; in how many days would a man working with a wheel and axle, raise the material, supposing a cubic foot to weigh 100 lbs.?

The area of a circle is found by squaring the diameter and multiplying by ·7854: 6 squared $6^2=36$.

Number of cubic feet in the shaft,

$$=6^2\times\cdot 7854\times 120=3392\cdot 928;$$

Weight of the material,

$$=3392\cdot 928\times 100=339292\cdot 8\text{ lbs.}$$

The units of work to raise this material $\frac{120}{2}$ feet,

$$=339292\cdot 8\times\frac{120}{2}=\mathbf{20357568};$$

for it is clear that the centre of gravity of the shaft is 60 feet from the top.

A man will perform **2600′** working 8 hours a day, turning a handle.

Units of work in 8 hours,

$$=2600\times 60\times 8=1248000.$$

The number of days required,

$$=\frac{20357568}{1248000}=16\cdot 33.$$

Ques. 10. Required the work in raising 3 cwt. of coals from a pit whose depth is 120 feet, the circumference of the rope being 2 inches, allowing the weight of 1 foot of the rope of 1 inch in circumference to be ·046 lbs.?

The weights of ropes of the same material, and of the same length, are as the squares of their circumferences.

Weight of one foot of rope,

$$=2^2\times\cdot 046=\cdot 184 \text{ lbs.}$$

Weight of the whole rope,

$$=\cdot 184\times 120=22\cdot 08 \text{ lbs.}$$

Units of work expended in raising the rope,

$$=22\cdot 08\times\frac{120}{2}=1324\cdot 8.$$

Work in raising the coals,

$$=112\times 3\times 120=40320;$$

Total work,

$$=40320+1324\cdot 8=41644\cdot 8.$$

Ques. 11. A cistern 22 feet long, 10 feet broad, and 8 feet deep; required the work in filling it, when the height of the bottom of the cistern from the water in the well is 36 feet?

Taking 1000 oz. or $62\frac{1}{2}$ lbs. as the weight of a cubic foot of water, the water in the cistern,

$$=22\times 10\times 8\times 62\cdot 5=110000 \text{ lbs.}$$

The height to which the centre of gravity of the water has to be raised,

$$=\frac{8}{2}+36=40 \text{ feet.}$$

$$\text{Work}=110000\times 40=4400000.$$

Ques. 12. The side A B of a cube of granite is 6 feet, and the weight of a cubic foot is 170 lbs.; it is required to find the work necessary to overturn it on the edge of A?

The distance of the centre of gravity, g, from the edge, A,

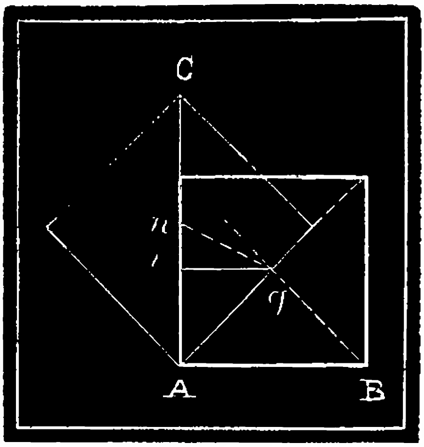
$$= \sqrt{\frac{6^2}{2}} = 4{\cdot}24 \text{ feet.}$$

When the centre is about to fall, the centre of gravity is raised from r to n, consequently the work in overturning the body is the same as raising its whole weight a perpendicular height $= r\,n$,

$$\therefore \quad r\,n = 4{\cdot}24 - 3 = 1{\cdot}24.$$

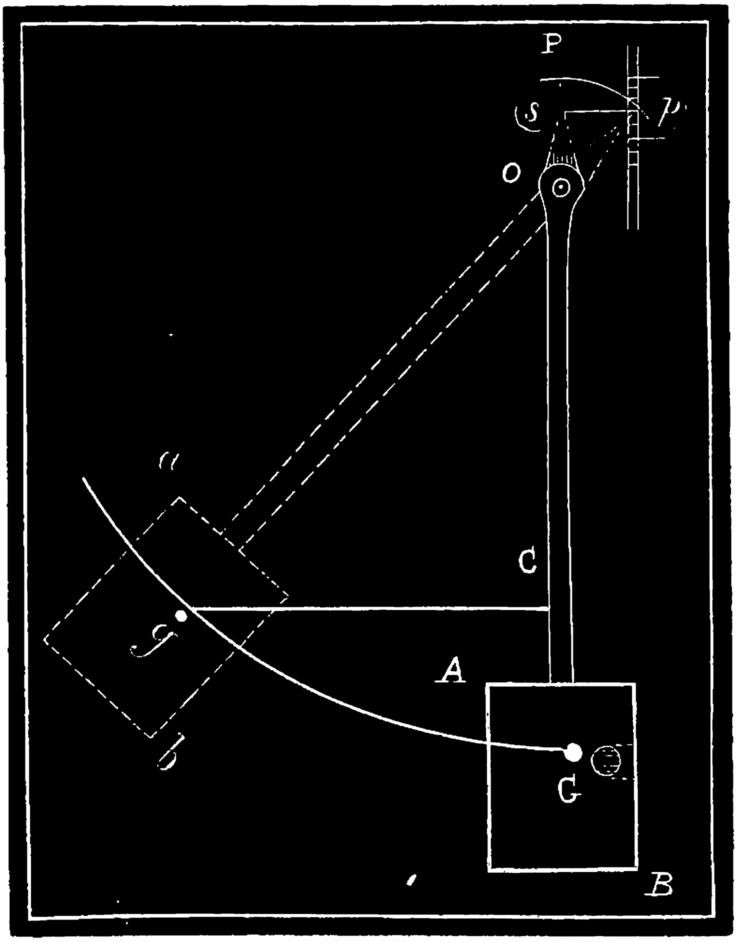
$$\text{Work} = 170 \times 6^3 \times 1{\cdot}24 = 45532{\cdot}8.$$

Fig. 4.



In a body of any form about to fall, the centre of gravity, g, will be at n in the vertical line, A C, the work in bringing the body to this position is due to the vertical distance, $r\,n$, through which the centre of gravity has been raised. The work necessary to overturn any body is the true measure of stability.

Fig. 5.



Ques. 13. Let a round shot strike a heavy pendulum block, A B, Fig. 5, and move with it to the position *a b*; let O be the point of suspension, G the centre of gravity of the block, shot, and pendulum taken together; O G = 17 feet $5\frac{1}{2}$ inches; O P = 12 inches, and the point *p* to be $2\frac{3}{4}$ inches lower than P, or *s* P = $2\frac{3}{4}$ inches. Find the units of work done by this round shot; the weight of the compound body put in motion being 2 tons 4 cwt. 25 lbs.; friction and the resistance of the air being neglected,

O P : Ps :: O G : G C, the

distance through which the centre of gravity, G, is raised —that is,

12 : $2\frac{3}{4}$ :: 17 feet $5\frac{1}{2}$ inches : 4 feet nearly.

3 tons 4 cwt. 25 lbs. = 7193 lbs.

7193 × 4 = **28772**, the units of work required.

CHAPTER IV.

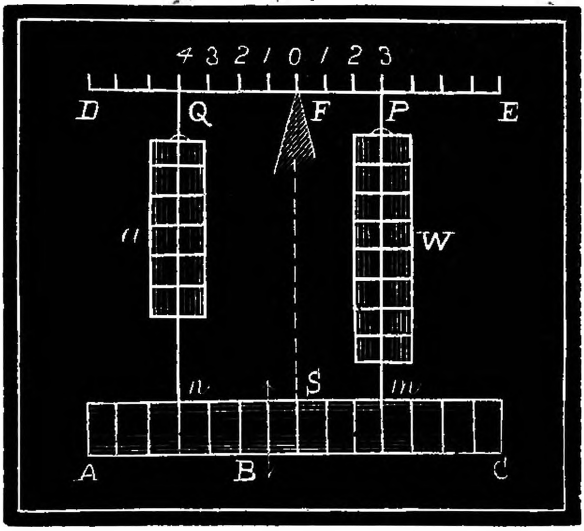
ON THE LEVER, WHEEL AND PULLEY, INCLINED PLANE, WEDGE AND SCREW.

In this chapter I intend to treat of the transmission of work by simple machines; it may be observed, that the object of machinery, whether simple or compound, is to regulate the distribution of work, or to change the direction—and not to increase work.

If the parts of a machine were not subject to friction or any other resistances, the *work* given out would be exactly equal to the work applied. Dead matter, by its gravity, produces pressure, and, by the intervention of mechanism, that pressure may be increased or decreased; but *work* is peculiarly the production of active or living agents.

THE LEVER.

Fig. 6.



Suppose two uniform bars, A B, B C, to be suspended from their centres, *n* and *m*, by means of cords attached to the points 4 and 3 of the lever D E turning on the fulcrum F, which must evidently be in the middle of D E = A C, and over the middle of A C, in order to secure equilibrium; that is, in order that the two parts may rest horizontally as if they were in one piece, and suspended by the middle point, S. It is also evident that the equilibrium will not be destroyed if the bars be hung by their ends at the points 4 and 3.

Let the weight of the bar, A B = 6 lbs., and the weight of B C = 8 lbs., then A B will contain 6 units of length, and B C, 8 units of length. It is evident from the figure that F q, the distance at which A B acts from the fulcrum, contains 4 units, and F p, the distance at which B C acts from the fulcrum, will contain 3 units; then since it appears that a weight of 6 lbs., suspended at q, balances a weight of 8 lbs., suspended at p; therefore the following relation exists when equilibrium takes place:

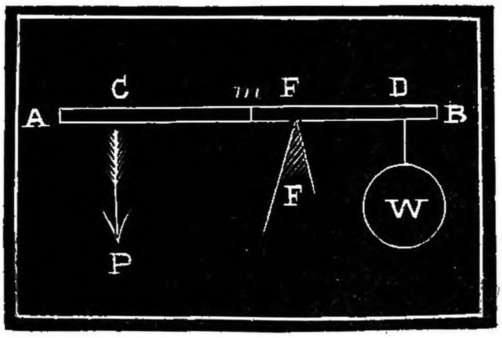
$$6 \text{ lbs.} \times 4 = 8 \text{ lbs.} = 3, \text{ generally}$$

$$w \times q\,F = W \times F\,p.$$

Levers are divided into three kinds: In the *first kind of lever*, the power and weight are on opposite sides of the fulcrum.

FIRST ORDER OF LEVER.

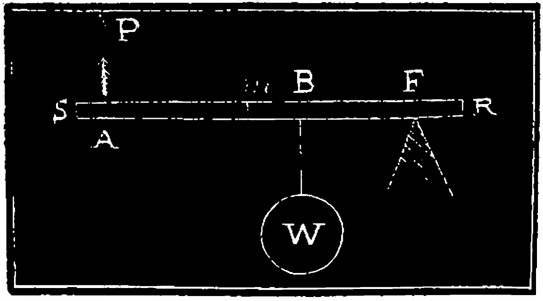
Fig. 7.



A B is a lever of the first order, F the fulcrum or prop, P the power acting at C, and W the weight attached to the point D. If CF be twice D F, 5 lbs. at C will balance 10 lbs. at D; generally as many times as C F is longer than F D, so many times will W be greater than P.

SECOND ORDER OF LEVER.

Fig. 8.



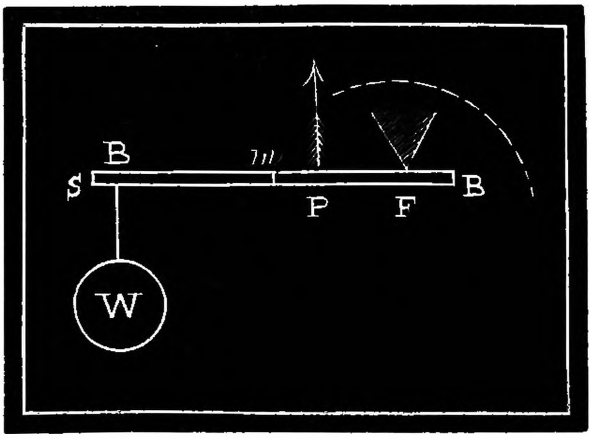
Lever of the second kind.—The weight is between the fulcrum and the power. W is the weight, F the fulcrum, P the power.

When the lever is supposed to be without weight, then if the length of A F = a, and B F = p, the power P balances the weight W, when

$$W \times p = P \times a.$$

THIRD ORDER OF LEVER.

Fig. 9.



Third kind or order of lever.—In this case the power is between the fulcrum and the weight. P represents the power, F the fulcrum, and W the weight. Generally if $BF = p$, and $PF = q$, the power P balances the weight W, when

$$W \times p = P \times q.$$

EXAMPLES.

Question 1. In a lever of the first kind, Fig. 7, let W = 30 lbs. The arm C F = 11 feet, the arm F D = 4; required P?

Put $x = P$ in lbs.

Then by the equality of moments,

$$11 \times x = 4 \times 30,$$

$$\therefore \quad x = \frac{120}{11} = 10\tfrac{10}{11} \text{ lbs.}$$

Ques. 2. A man exerts a pressure of 50 lbs. on a crowbar at a distance of 4 feet from the fulcrum, what weight will he balance at the distance of 3 inches from the fulcrum?

$$W \times 3 = 50 \times 48,$$

$$\therefore \quad W = 800 \text{ lbs.}$$

Ques. 3. In a lever of the second kind, Fig. 8, W = 11 lbs., B F = 16 inches, and A F = 100 inches; required P?

$$P \times 100 = 11 \times 16,$$

$$\therefore \quad P = 1{\cdot}76 \text{ lbs.}$$

Ques. 4. In a lever of the third kind, Fig. 9, W = 40; B F = 60 inches, and P F = 8; required P?

$$P \times 8 = 40 \times 60,$$

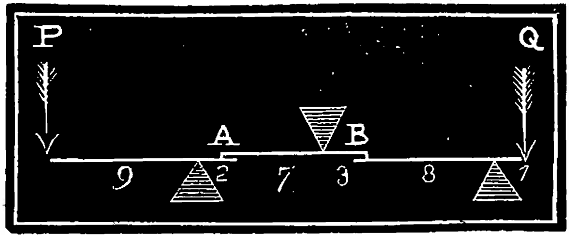
$$\therefore \quad P = 300 \text{ lbs.}$$

Ques. 5. In a lever of the first kind, 5 and 8 lbs. are placed on one side of the fulcrum, at a distance of 6 and 4 inches respectively from the fulcrum; required the power P, acting at the distance of 9 inches from the fulcrum, to maintain equilibrium?

$$P \times 9 = 5 \times 6 + 8 \times 4,$$

$$\therefore \quad P = \frac{62}{9} = 6\frac{8}{9} \text{ lbs.}$$

Fig. 10.



Ques. 6. In a combination of three levers of the first order, represented on the accompanying diagram, Fig. 10, the long arms are 9, 7, 8, inches respectively, the short arms 2, 3, 1 inches; if a pressure of 10 lbs. be applied at P, what is the pressure at Q to balance it?

$9 \times 10 = 2 \times$ the pressure at A,

The pressure at $A = 45$ lbs.,

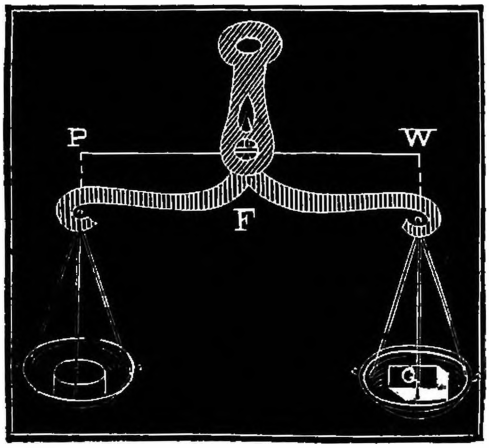
$7 \times 45 = 3 \times$ the pressure at B,

Pressure at $B = 105$ lbs.

$105 \times 8 = 1 \times$ the pressure at Q.

Pressure at $Q = 840$ lbs.

Fig. 11.



Ques. 7. Let P F and F W be the arms of a false balance, a certain weight, Q, weighs 16 lbs. when put in the scale attached to the long arm, and only 9 lbs. when weighed in the opposite scale; what is the true weight of Q?

By the equality of moments, the two following equations are obtained:

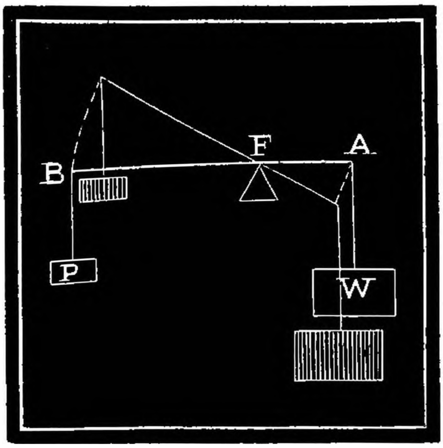
$$Q \times W\,F = 9 \times P\,F;$$
$$Q \times P\,F = 16 \times W\,F.$$

Multiplying these equations together, and then striking out the common factors,

$$Q \times Q = 9 \times 16;$$
$$\therefore \quad Q = 12.$$

THE PRINCIPLE OF WORK APPLIED TO THE LEVER.

Fig. 12.



Let B F = 10 feet, F A = 2 feet, and P = 3 lbs.; required W?

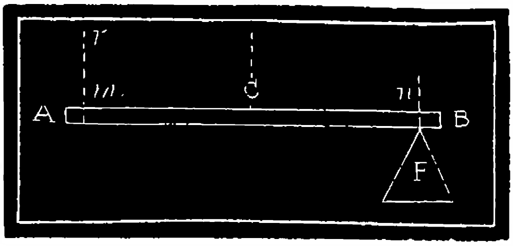
It is evident, if P be raised 5 feet, W will be depressed one, because F B is five times the length of F A; consequently the work of P = $\mathbf{3} \times \mathbf{5}$, and the work of W = W × 1;

$$\therefore \quad W \times 1 = \mathbf{3} \times \mathbf{5};$$
$$\therefore \quad W = 15 \text{ lbs.}$$

Hence the equality of moments is readily established by the principle of work, and conversely. When the motion is extremely small, the principle of work is termed the principle of virtual velocities.

OF THE LEVER WHEN ITS WEIGHT IS TAKEN INTO ACCOUNT.

Fig. 13.



The tendency of a uniform beam or lever, A B, to turn about the fulcrum, F, is just the same as if the whole of its weight were collected in its middle point, or centre of gravity, C. For if the preponderating side, A F, be hung from a cord, $m\ r$, placed so that A $m = n$ B, then this cord will sustain one-half the weight of the beam, and the fulcrum, F, the other half. Now, if the whole weight of the beam be collected at C, the centre, it would produce the same strain upon the cord, hence the beam acts as if its whole weight were collected in its middle point.

Ques. 8. The weight of the lever of the first kind=10 lbs., Fig. 7, the length A B=56 inches; A F=40; C F=36; D F=5 inches; and W=150 lbs.; required P, in order to maintain the lever in equilibrium? In this case the weight of the lever acting at its centre of gravity, m, co-operates with the power,

$$m\,\mathrm{F} = \mathrm{A\ F} - \mathrm{A}\ m,$$

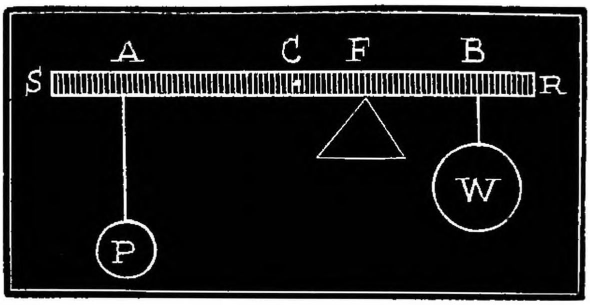
$$= 40 - \tfrac{1}{2} \text{ of } 56 = 12$$

Then by the equality of moments,

$$\mathrm{P} \times 36 + 10 \times 12 = 5 \times 150.$$

$$\mathrm{P} = 17{\cdot}25 \text{ lbs.}$$

Fig. 14.



Ques. 9. The weight of the lever S R, Fig. 14, of the first order=31 lbs.; S R=85 inches; S F=55; A F=43; B F=19; and W=34 lbs. Required P?

$$\mathrm{S\ F} = 55.$$

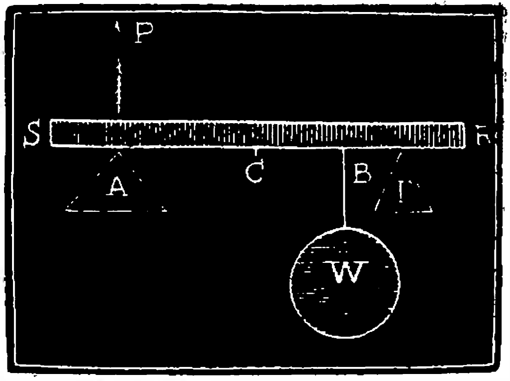
$$\mathrm{S\ C} = \frac{85}{2} = \underline{42\tfrac{1}{2}}$$

$$\mathrm{C\ F} = \underline{11\tfrac{1}{2}};$$

$$\therefore\quad \mathrm{P} \times 43 + 31 \times 11\tfrac{1}{2} = \mathrm{W} \times 19,$$

$$\therefore\quad \mathrm{P} = \frac{34 \times 19 - 31 \times 11\frac{1}{2}}{43} = 6\tfrac{63}{86} \text{ lbs.}$$

Fig. 15.



Ques. 10. The weight of a lever, Fig. 15, of the second order=8 lbs. S R=80 inches; S F=76; A F=70 inches; B F=2 inches, and P=100 lbs. required W?

In this case, the weight of the lever acts with W;

$$\therefore \quad 76 - \frac{80}{2} = 36 = \mathrm{C\,F}.$$

$$W \times 2 + 8 \times 36 = 100 \times 70,$$

$$\therefore \quad W = 3356 \text{ lbs.}$$

Ques. 11. A beam, R S, Fig. 15, whose weight is 4 cwt., is supported by props at A and F; a weight, W, of 20 cwt. is placed at B; it is required to determine the pressures on the props, when R S=50 feet; F R=2 feet; B F=12 feet; A F=30 feet?

Let C be the centre of the beam, then C R=$\frac{1}{2}$ of 50=25; C F=25—2=23; suppose the beam to turn on F as a fulcrum, the pressure on A,

$$= P \times 30 = 20 \times 12 + 4 \times 23,$$

$$\therefore \quad P = \frac{332}{30} = 11\frac{2}{30} \text{ cwt.}$$

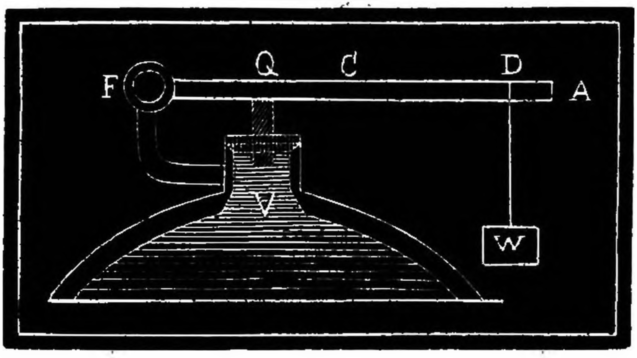
Because the two props support the whole weight 24 cwt., $24 - 11\frac{2}{30} = 12\frac{14}{15}$ cwt. the pressure on F.

HOW TO GRADUATE THE LEVER OF A SAFETY VALVE OF A STEAM ENGINE.

A F is a graduated lever of the second kind, turning on F as a centre; V the valve of opening or closing the communication of steam in the boiler with the atmosphere; the lever, A F, rests upon the pin, Q V, of the valve, V, and a sliding weight, W, is suspended from the lever, enabling the engineer to place different amounts of pressure on the valve; this pressure measures the elasticity of the steam when it begins to escape. When steam of very high temperature is employed, the admission of atmospheric air is dangerous, and under such circumstances this valve may be properly termed the unsafety valve. In order to graduate the lever, a weight, W, must be found, so that when it is placed at A it may balance the greatest pressure of the steam.

The next thing to be found, is the position of the weight W, to give any proposed pressure to the valve.

Fig. 16.



EXAMPLES.

Ques. 12. The length A F of a lever = 14 inches; the distance F Q = 2 inches; the weight of the valve and pin = 5 lbs.; the weight of the lever F A = 8 lbs., and the area of circular valve in the narrowest place = 6 square inches; it is required to find the load W, so that when it is placed at the extremity of the lever, the steam may have a pressure of 30 lbs. on the square inch, or 45 lbs. including the pressure of the atmosphere?

Pressure of steam on valve,

$$= 6 \times 30 = 180 \text{ lbs.}$$

Hence the effective pressure on the lever,

$$= 180 - 5 = 175 \text{ lbs.},$$

since the weight of the lever will act at the middle point, C,

$$W \times 14 + 8 \times 7 = 175 \times 2$$

$$\therefore \quad W = 21 \text{ lbs.}$$

Ques. 13. At what distance from the fulcrum must the load be, in the last example, so that the steam may have a pressure of 16 lbs. over the pressure of the atmosphere?

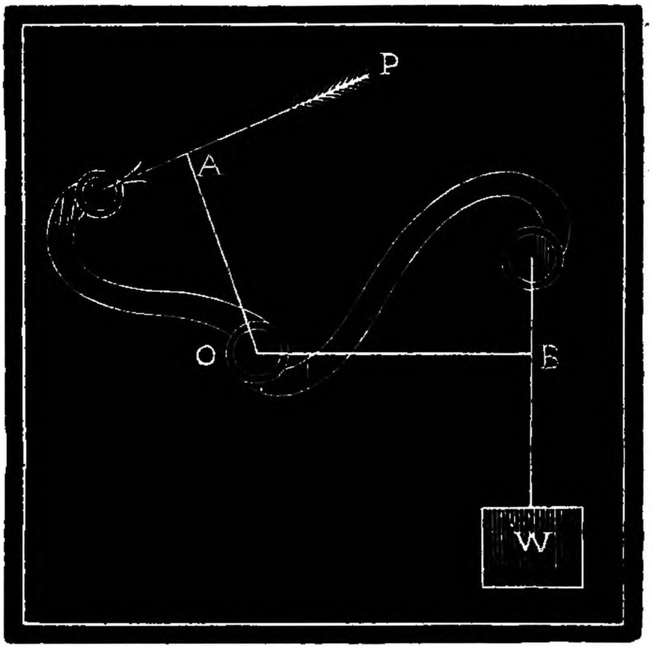
$$16 \times 6 - 5 = 91.$$

Suppose D, Fig. 16, to be the required position of the load, then

$$F D \times 21 + 8 \times 7 = 91 \times 2;$$
$$\therefore \quad F D = 6 \text{ inches}.$$

Ques. 14. In a bent lever, Fig. 17, the perpendicular, O A, on the direction of the force, P, is 4 inches; and the perpendicular, O B, upon the direction of W, is 7 inches; required P when W = 200 lbs.?

Fig. 17.

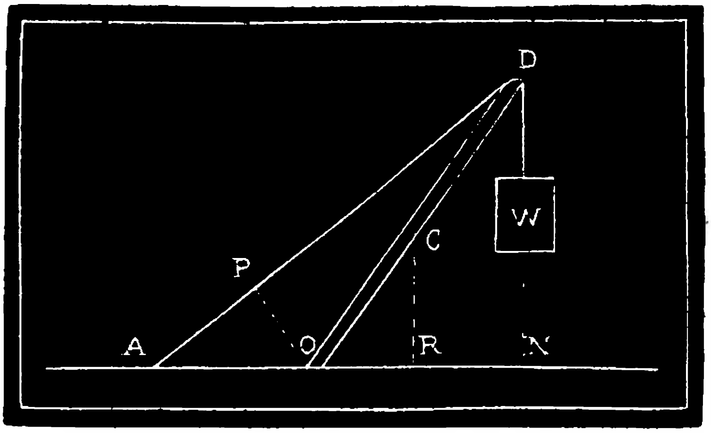


$$P \times 4 = 200 \times 7,$$
$$\therefore \quad P = 350.$$

Ques. 15. A rope, A D, Fig. 18, supports a uniform pole, O D, resting on the ground at O, and supporting the weight, W, suspended from D; required the tension of the rope, when A D = 175 feet, O A = 40 feet, O D = 145 feet, W = 50 cwt., and weight of O D = 10 cwt.?

In this example O D may be regarded as a lever turning on O as a centre. Draw O P perpendicular to A D, and C R, a vertical line, passing through the centre of

Fig. 18.



gravity, C, of the pole, then the moment of the force stretching the rope will be equal to the sum of the moments of W and the weight of the pole; that is,

$$\text{Tension of cord} \times OP = W \times ON + \text{Wt. of pole} \times OR.$$

To find the perpendiculars,

$$OP,\ DN,\ CR,$$

in order to do this in the most simple manner, it will be observed, that since the three sides of the triangle, A O D, are given, the area may be found by the common rule; the area = 2100 square feet; but the area is also expressed by

$$\frac{AD \times OP}{2} = 2100,$$

$$\therefore \quad OP = 24 \text{ feet.}$$

But the area is also =

$$\frac{AO \times DN}{2} = 2100,$$

$$\therefore \quad DN = 105 \text{ feet.}$$

$$ON = \sqrt{145^2 - 105^2} - 100,$$

$$\therefore \quad OR = 50.$$

$$\therefore \quad \text{Tension of cord} \times 24 = 50 \times 10 + 100 \times 50.$$

Consequently the tension of the cord

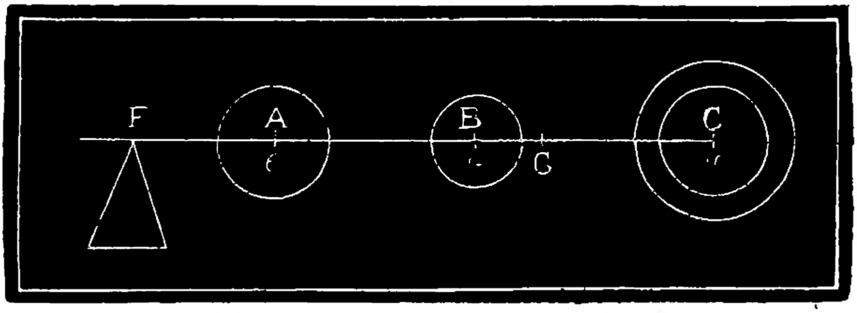
$$-229\tfrac{1}{6} \text{ cwt.}$$

THE CENTRE OF GRAVITY.

Without involving error, we may suppose a body acted upon by the earth's gravity to be composed of an infinite number of particles, each of which is acted on in a direction perpendicular to the horizon. The sum of all the parallel forces is evidently the weight of the body.

There is a point, where a single force, equal to the weight of the body, being applied, will produce the same effect as the force of gravity acting upon the various particles composing the body; this point is called the centre of gravity of the body. From this definition, it immediately follows that if the centre of gravity be supported, the body will stand, and *vice versâ.* But it must not be forgotten that a heavy body may be moved with ease on an axis passing through and supporting its centre of gravity. That the centre of gravity of all symmetrical or regular bodies is in their centre of magnitude. That if any body, acted on by the force of gravity, tend to turn a lever, we may regard the weight of the whole body to be collected in its centre of gravity.

Fig. 19.



Ques. 16. Let A, B, and C, be three bodies in the same right line, it is required to determine the position of their common centre of gravity, G, with respect to any assumed point, F, when $A=6$ lbs., $B=4$ lbs., $C=10$ lbs.; $AF=10$ feet, $BF=25$ feet, and $CF=42$ feet?

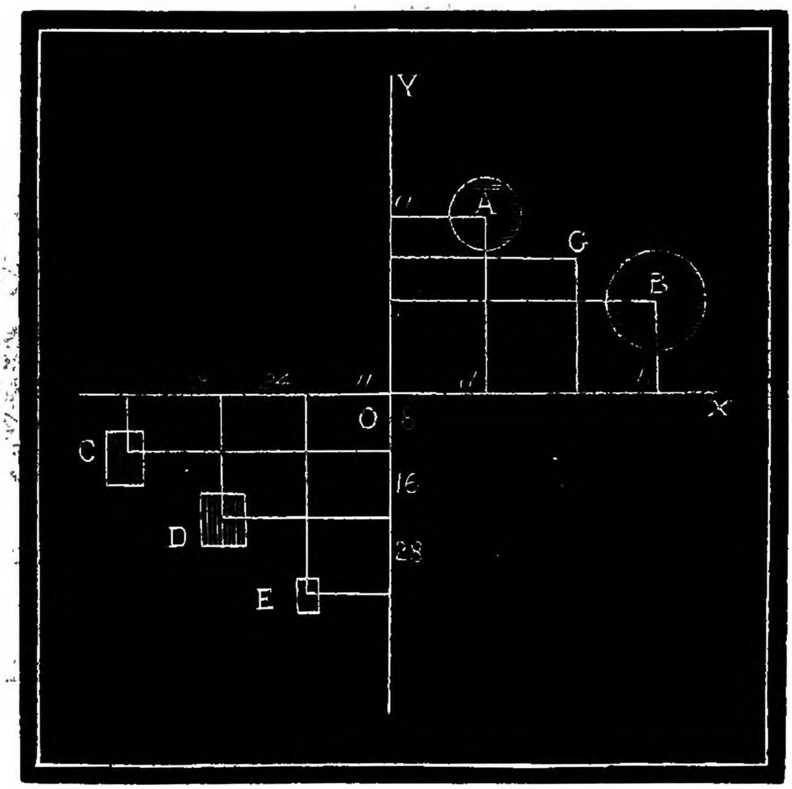
Let it be supposed that an inflexible rod, without weight, passes through the bodies, and that the system turns upon F as the fulcrum; then as the whole mass may be regarded as acting in the point G, by the equality of moments.

$$\text{F G} \times (6+4+10) = 6 \times 10 + 4 \times 25 + 10 \times 42,$$
$$\therefore \quad \text{F G} \times 20 = 580,$$
$$\text{F G} = 29 \text{ feet.}$$

Let A, B, &c., being any number of bodies lying in the same horizontal plane; it is required to determine the position of the centre of gravity, G, referred to two axes or lines, O X, O Y, perpendicular to each other, these lines, O X and O Y, are termed co-ordinate axes.

Conceive the bodies to be connected with the axis, O X, by the perpendicular rods, A *d*, B *b*, &c., then the sum of the moments of the bodies, tending to turn round the axis, O X, will be the same as the moment of the whole mass, collected in the centre of gravity, G. From this equality the distance of G from O X is obtained. In precisely the same way, we find the distance of G from O Y; and hence

Fig. 20.



the point G becomes known. After the same manner the position of the centre of gravity may be found, when the bodies are in space, by referring them to co-ordinate planes.

Ques. 17. The weights of three bodies, C, D, E, are 10, 17, and 9 lbs. respectively (Fig. 20); and their distances from O X are 8, 16, and 28 inches, and from O Y, 38, 24, and 11 inches respectively; required the position of the common centre of gravity?

Let x be the distance of the centre of gravity from the axis, O X; and y the distance from O Y, then

$$(10+17+9)\,x=10\times8+17\times16+9\times28,$$

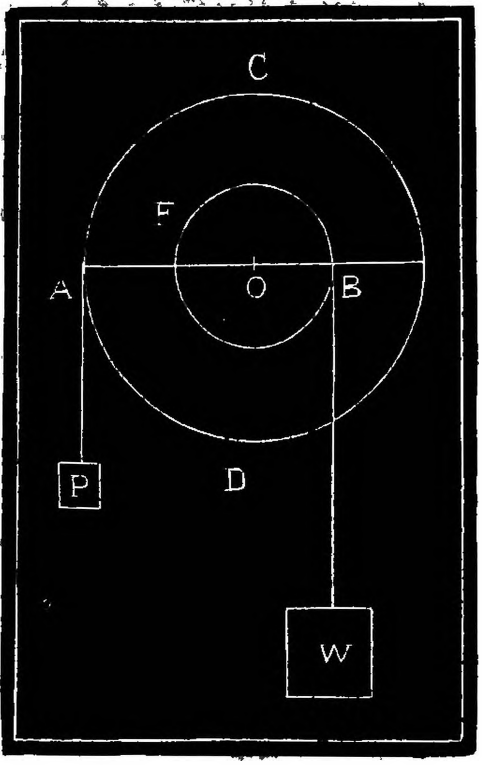
$$\therefore\quad x=16\tfrac{7}{9},$$

$$(10+17+9)\,y=10\times38+17\times24+9\times11,$$

$$\therefore\quad y=24\frac{23}{36}.$$

WHEEL AND AXLE.

Fig. 21.



This simple machine is only another form of the lever where the power is made to act continuously; it consists of a large wheel, A C D, and a cylinder, or axle, B F, both of which turn on the same axis, O. If the wheel, A C D, be turned round by a power, P, applied to it, the axle, B F, will coil up the rope by which the weight W hangs. The lever by which P acts is evidently A O, and that by which W acts is O B; hence when these weights, or pressures, balance each other,

$$P\times AO=W\times OB.$$

If the wheel be displaced by a handle, then the machine is termed a *windlass*. Sometimes the handle is made to turn a series of wheels acting on each other by means of teeth; a machine so formed is called a *crane*.

I will now consider the equilibrium, on the principle of work.

When the diameter of a circle $=1$, the circumference $=3{\cdot}1416$.

When the wheel makes 1 revolution, the axle also makes one. In one turn P descends a space,

$$=2\times OA\times 3{\cdot}1416,$$

and W ascends a space

$$=2\times OB\times 3{\cdot}1416.$$

The work done by P in one revolution

$$=\mathbf{2\times AO\times 3{\cdot}1416\times P};$$

and the work of W in one revolution

$$=\mathbf{2\times OB\times 3{\cdot}1416\times W}.$$

It is clear that the work done in one revolution is equal to the work applied, friction being neglected,

$$\therefore\ \mathbf{2\times AO\times 3{\cdot}1416\times P=2\times OB\times 3{\cdot}1416\times W}.$$

$$\therefore\ P\times OA=W\times OB,$$

which is the relation already established. Conversely, if we assume the equation of equilibrium, the principle of work may be readily established. It must be remembered, in applying the principle of work, that O A, O B, are taken in feet or decimal parts of a foot, and that W and P are measured in lbs.

Ques. 18. The handle of a windlass is 18 inches, the radius of the axle $=3$ inches, and the power applied $=60$ lbs.; what weight could be raised, friction being neglected?

Work of P in one revolution,

$$=\mathbf{60\times\frac{2\times 18}{12}\times 3{\cdot}1416}.$$

Work of W in one revolution,

$$=W\times\mathbf{\frac{2\times 3}{12}\times 3{\cdot}1416}.$$

$$\therefore\ W\times\mathbf{\frac{2\times 3}{12}\times 3{\cdot}1416=60\times\frac{2\times 18}{12}\times 3{\cdot}1416}.$$

$$\therefore\ W=360\text{ lbs.}$$

Ques. 19. Required W in the last example, when the thickness of the cord is 1 inch, supposing that $\frac{1}{8}$ of the work applied to be lost in friction, and the rigidity of the cord?

The cord increases the radius of the wheel $\frac{1}{2}$ of an inch, hence

Work of W in one revolution,

$$= W \times \frac{7}{12} \times 3{\cdot}1416;$$

Effective work of P in one revolution,

$$= \frac{7}{8} \times 60 \times 3 \times 3{\cdot}1416,$$

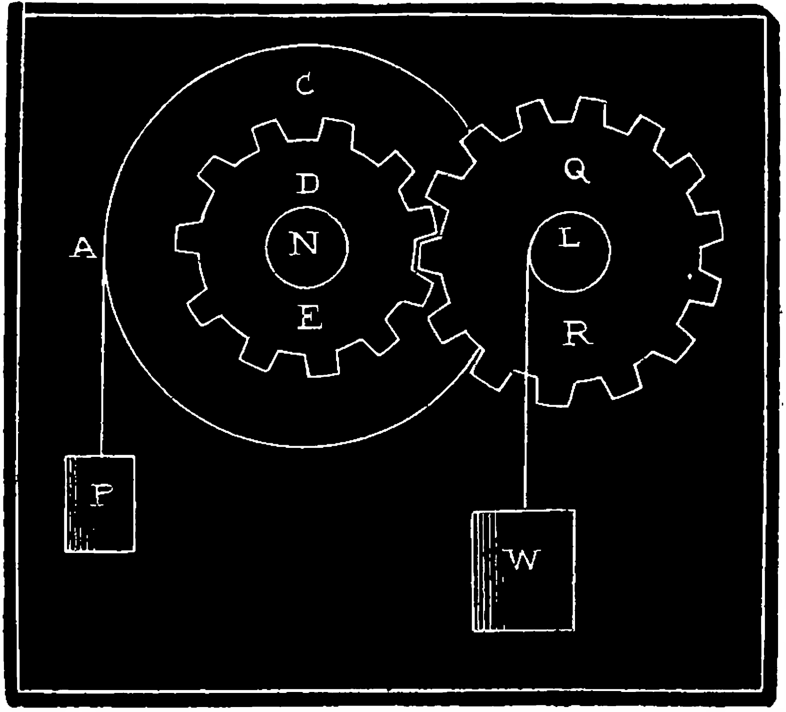
$$\therefore \quad W \times \frac{7}{12} \times 3{\cdot}1416 = \frac{7}{8} \times 60 \times 3 \times 3{\cdot}1416.$$

$$\therefore \quad W = 270 \text{ lbs.}$$

OF COGGED OR TOOTHED WHEELS.

Suppose the cogged wheel D E to turn upon the same axis, N, as the wheel C; Q R another cogged wheel, acted

Fig. 22.



upon by the former and turning upon the same axis as the axle L. From the wheel C is suspended the weight P, and from L the weight W; then while P descends, the wheel C and the cogged wheel D E will be turned from right to left, but as each tooth in the cog-wheel D E is being turned round, a corresponding tooth in the cog-wheel Q R will be turned in the contrary direction, and thus the cord L W will be coiled up on the axle L, and the weight W raised.

Ques. 20. Let $P = 120$ lbs., the diameter of the wheel C. $= 3$ feet, the number of teeth in D $E = 11$, the number in Q $R = 14$, the diameter of the axle $L = \frac{1}{2}$ a foot; required W, in order that equilibrium may take place, friction being neglected?

For every turn of the wheel C, 11 teeth of the wheel Q will be turned round, therefore as many times as the number of teeth in D can be taken out of the number in Q, so many turns will the cog-wheel D E make while the wheel Q R makes one.

Let the axle L and wheel Q R make one revolution, then the revolutions made by the wheel C,

$$= 14 \text{ divided by } 11 = \tfrac{14}{11}.$$

The space moved over by W,

$$= \tfrac{1}{2} \times 3{\cdot}1416;$$

The space moved over by P,

$$\frac{14}{11} \times 3 \times 3{\cdot}1416;$$

$$\text{Work due to } W = \frac{1}{2} \times 3{\cdot}1446 \times W.$$

$$\text{Work due to } P = 3 \times 3{\cdot}1416 \times \frac{14}{11} \times 120.$$

∴ Since the work due to P and W are equal during one revolution of L, neglecting friction,

$$\frac{1}{2} \times 3{\cdot}1416 \times W = \frac{14}{11} \times 3 \times 3{\cdot}1416 \times 120,$$

$$\therefore\ \tfrac{1}{2} W = \tfrac{14}{11} \times 3 \times 120.$$

$$W = 916\,\tfrac{4}{11} \text{ lbs.}$$

Ques. 21. Let the handle of a crane, Fig. 22, describe the circle A C, whose radius=19 inches, and suppose 12 the number of teeth in D E, and 60 in Q R; the diameter of the axle L=$4\frac{3}{4}$ inches; it has been found that 200 lbs. pressure applied to the handle will only raise 5000 lbs.; what is lost by friction?

Feet passed over by W in one revolution of Q R,

$$=\frac{4\frac{3}{4}\times 3{\cdot}1416}{12};\text{ and}$$

because $\frac{60}{12}=5$, the space passed over by P for one revolution of Q R$=\frac{38\times 3{\cdot}1416\times 5}{12}$ feet.

∴ Units of work due to P=

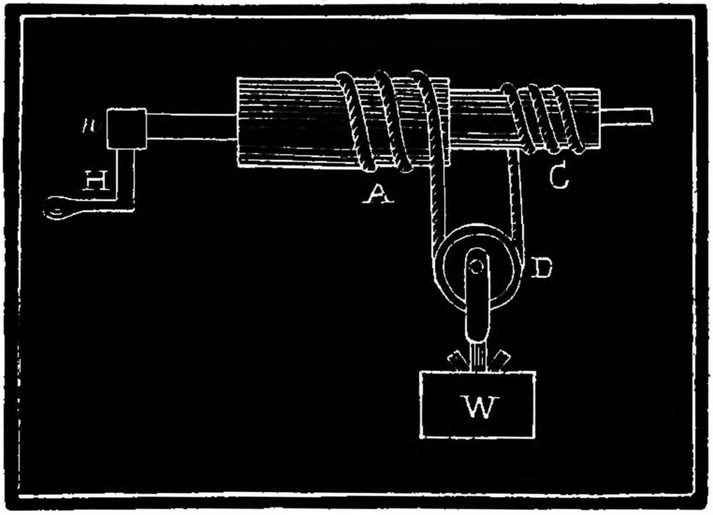
$$\frac{38\times 3{\cdot}1416\times 5}{12}\times 200=\frac{4\frac{3}{4}\times 3{\cdot}1416}{12}\times W,$$

the friction being neglected, in this case W = 8000 lbs. Hence 3000 lbs. is due to the friction of the machine, $\frac{3}{8}$ of the power being lost.

COMPOUND WHEEL AND AXLE.

In the ordinary wheel and axle, there is a practical limit to the power of the machine; for we can only increase the

Fig. 23.



power by increasing the size of the wheel, or by decreasing the radius of the axle. But in the compound wheel and axle, a given power may be made to raise very great weights.

The machine consists of two axles, A and C, cut upon the same block, round which a cord coils in opposite directions; this cord passes round the movable pulley D, which carries the weight W. Now, if the handle be turned, one of the cords is coiled upon the large axle A, while the other cord is uncoiled from the small axle C, so that the rate at which W ascends depends upon the difference of the circumferences of the two axles; and, consequently, the power of the machine will also depend upon this difference, which may be decreased to any extent, without altering the length of the handle. This ingenious contrivance is due to the Chinese.

Ques. 22. Let the diameter of the axle A = ·8 feet, the diameter of axle C = ·6 feet, the length of the handle Hn = 3½ feet, and W = 6090 lbs. Required P?

When the handle Hn, Fig. 23, is turned once round, the cord A will be drawn up a space equal to the circumference of the axle A, while the cord C will be let down a space equal to the circumference of the axle C; therefore the whole cord will be shortened a space equal to the difference of the circumferences, and, because the cord is doubled, the weight W will be raised a space only equal to the half of this difference.

Units of work expended on W in one revolution of Hn =

$$\frac{\cdot 8 \times 3\cdot 1416 - \cdot 6 \times 3\cdot 1416}{2} \times 6090;$$

Work done upon P, in one revolution,

$$= 7 \times 3\cdot 1416 \times P;$$

$$\therefore \quad 7 \times 3\cdot 1416 \times P = \frac{\cdot 8 \times 3\cdot 1416 - \cdot 6 \times 3\cdot 1416}{2} \times 6090$$

Dividing each side by 3·1416,

$$\therefore \quad 7P = \frac{\cdot 8 - \cdot 6}{2} \times 6090,$$

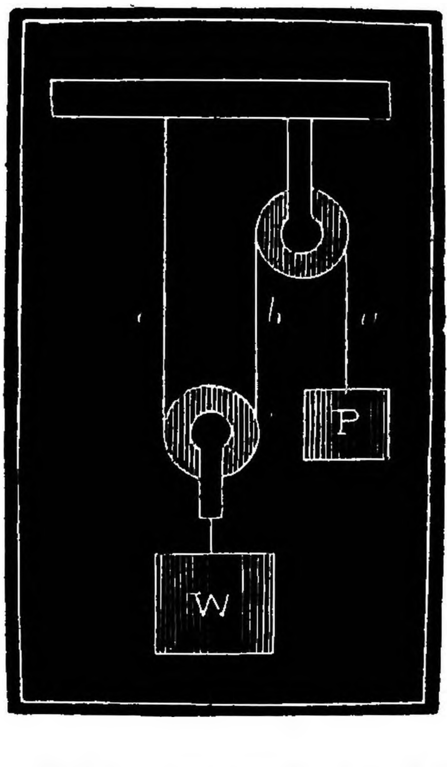
$$P = 87 \text{ lbs.}$$

THE PULLEY.

A pulley is a grooved wheel, turning on an axis, and placed in a block or case. A cord passes over the groove of the wheel, in order to transmit the force applied in any proposed direction. There is no advantage gained by a single fixed pulley, except in changing the direction of the force applied, which is diminished by the friction of the pulley; but, when there are movable pulleys, the weight raised will always be greater than the power applied; and then the advantage depends upon the number of cords by which the weight is suspended.

Ques. 23. In the annexed system of pulleys, if W=500 lbs., required P, when equilibrium takes place?

Fig. 24.



As W is suspended by two cords, *c* and *b*, each cord will support 250 lbs.; but as the cord is supposed to have a free motion over the wheels, the portions *a*, *b*, *c*, will have the same stretch or tension; hence P=250 lbs.

Application of the Principle of Work.

Let P and W be expressed in pounds, then if W, Fig. 24, ascends 1 foot, the cords *c* and *b* will each be shortened 1 foot, and therefore P must descend 2 feet. Hence the work of P

$$=P\times 2;$$

and the work of $W=W\times 1$;

$$\therefore\ P\times 2=W\times 1.$$

$$P=\frac{W}{2}.$$

Ques. 24. Let there be two movable pulleys, each weighing 4 lbs., then if P=60 lbs., required the weight raised independent of the weight of the pulleys, on the principle of work?

If P descend 4 feet, the first movable pulley will ascend 2 feet, and the second 1 foot.

Consequently, the work done in raising W and the pulleys

$$=W\times 1+4\times 1+4\times 2=W+12;$$

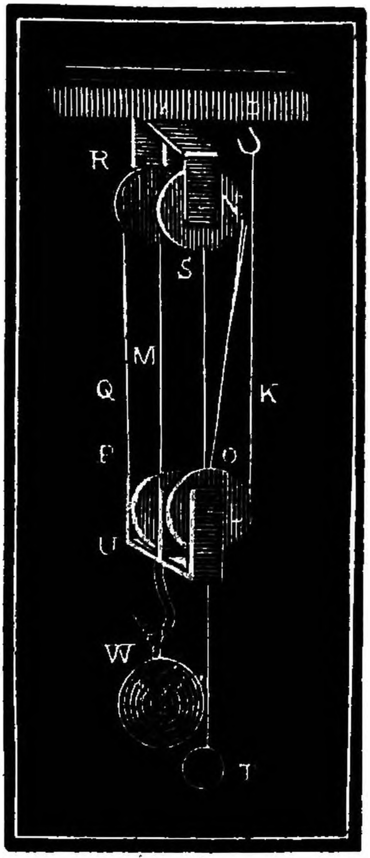
The work of $P=60\times 4$;

$$\therefore\ W+12=60\times 4.$$

$$W=228 \text{ lbs.}$$

$$\therefore\ \text{the weight}=220 \text{ lbs.}$$

Fig. 25.



Ques. 25. Let the cord I K L M N O P Q R S T, Fig. 25, made fast to the hook I, pass over two fixed pulleys, *u*, and the two movable pulleys, U, and a pressure of 200 lbs. applied at T was found by experiment to raise only a weight, W, of 565 lbs.; what part of the power is lost by friction, the rigidity of the cord, and the weight of the movable pulleys?

When the weight W is raised 1 foot, each of the cords I K, M, O N, and P R, are shortened 1 foot, and the cord S T lengthened 4 feet. Hence, when friction, &c., are neglected, the work done on W in raising it a foot $= \mathbf{W} \times \mathbf{1}$, which must be equal to the work of $\mathbf{T} = \mathbf{T} \times \mathbf{4}$, as T moves 4 feet, while W moves over 1.

$$\because \quad \mathbf{W} \times \mathbf{1} = \mathbf{T} \times \mathbf{4}.$$

$$\therefore \quad \mathrm{T} = \frac{\mathbf{W}}{4} = 141\tfrac{1}{4} \text{ lbs.}$$

in the present case.

$$\therefore \quad 200 - 141\tfrac{1}{4} = 58\tfrac{3}{4},$$

and $\dfrac{58\frac{3}{4}}{200} = \cdot 29375$ the part of the power lost.

OF THE INCLINED PLANE.

To find the pressure necessary to support a body on an inclined plane without friction. Let the weight W, Fig. 26, be drawn up the inclined plane, A C, by means of the weight P, acting by a cord parallel to the plane; then whilst W is moved from A to C, the weight P will have descended a vertical space equal to A C. Then, according to the principle of work,

Work in raising $W = W \times CB$;

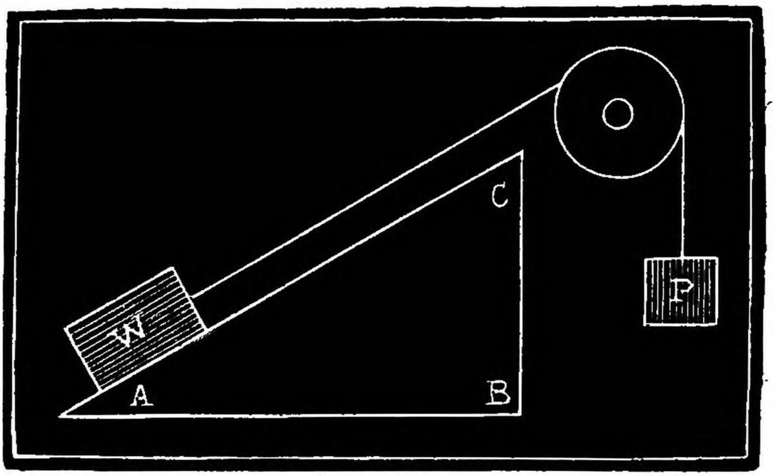
Work due to the descent of P

$$= P \times AC;$$

$$\therefore \quad P \times AC = W \times CB;$$

$$\therefore \quad P = \frac{CB}{AC} W.$$

Fig. 26.



Ques. 26. The length of an inclined plane is 30 feet, the perpendicular height 10 feet, and the weight of the body W = 20 cwt., what pressure, P, will be required to sustain the body on the plane, friction being neglected?

Work in raising W in opposition to gravity,

$$= 20 \times 112 \times 10 = 22400;$$

The work of $P = P \times 30$;

$$\therefore \quad P \times 30 = 22400.$$

$$P = 746\tfrac{2}{3} \text{ lbs.}$$

Ques. 27. The length of an inclined plane is a mile, or 5280 feet, the height 88 feet, the weight of the body 1760 lbs., and the friction $\frac{1}{264}$ part of the weight, what pressure will be required to move the body up the plane?

In this case the inclination of the plane being slight, the pressure upon it is very nearly equal to the weight of the body.

$\therefore$ Pressure to overcome friction$=\frac{1760}{264}$ lbs.

Work due to friction when the body is moved over one mile,$=$

$$\frac{1760}{264}\times 5280=35200;$$

Work due to gravity,$=$

$$1760\times 88=154880;$$

Work of the pressure, P, applied to move the body,

$$=P\times 5280;$$

$$\therefore\quad P\times 5280=35200+154880.$$

$$\therefore\quad P=36 \text{ lbs.}$$

OF THE WEDGE, OR MOVABLE INCLINED PLANE.

Let A B C be a wedge, Fig. 27, sliding on the horizontal plane A B, by the action of the pressure, P, applied parallel to A B, and thereby elevating the centre of gravity of a weight, W, in a vertical direction. When the wedge begins to act, the resistance, W, rests upon the horizontal plane A B; but when the wedge has moved over a space equal to its length, the point *q* will have been elevated a height equal to the thickness, C B, of the wedge, and the centre of gravity, G, some vertical height *m n*; then if A B$=1{\cdot}2$ feet, B C$={\cdot}2$ feet, and the pressure at $q=60$ lbs., not taking friction into account,

$$\text{Work applied by } P=P\times 1{\cdot}2.$$

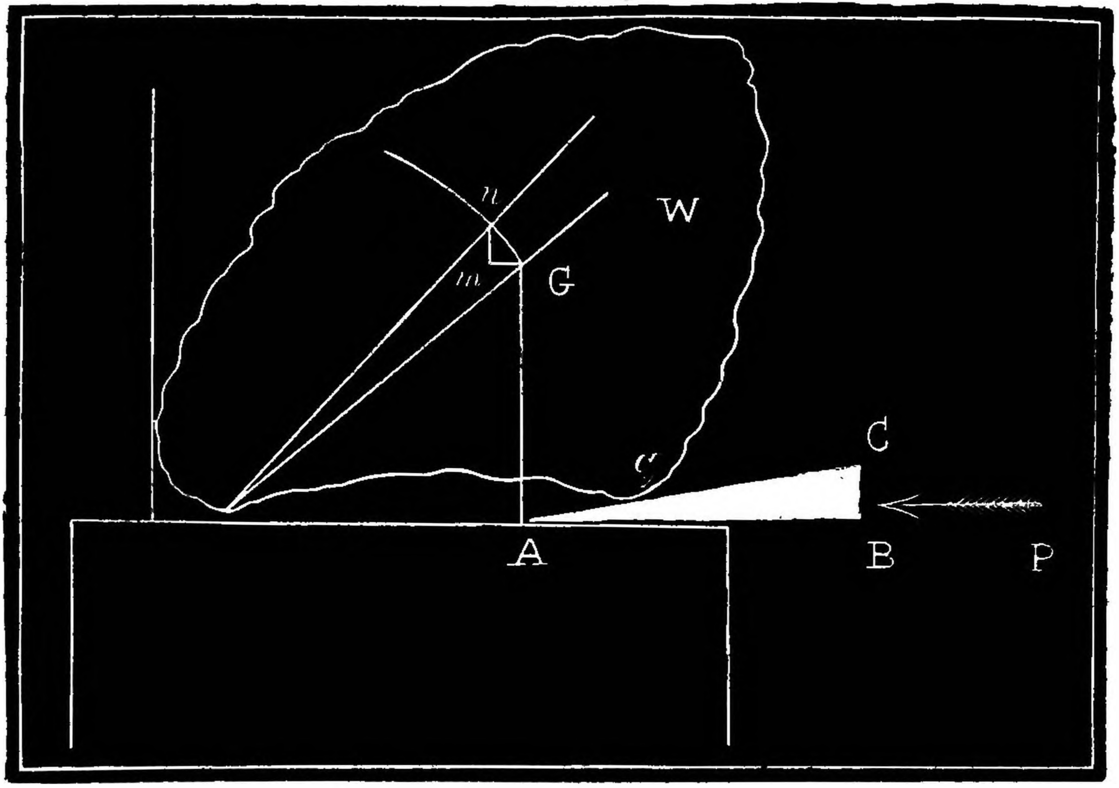
$$\text{Work of } q=60\times{\cdot}2;$$

$$\therefore\quad \times 1{\cdot}2=60\times{\cdot}2;$$

$$\therefore\quad P=10 \text{ lbs.}$$

This calculation shows that the advantage of the power depends upon the thinness of the back of the wedge. But it is necessary to observe, that the astonishing power of the wedge, as usually employed, is equally resolvable into the force of impact.

Fig. 27.



Ques. 28. With a velocity of 38·6 feet a second twenty blows of a sledge-hammer of 30 lbs. weight are delivered on the head of the wedge A B C, Fig. 27, and drives it 3 inches in the direction A B. W = 300 tons, its centre of gravity, G, moves from G to *n*, and is raised a vertical height *m n* = 1 inch when *q* is raised 2 inches; find what part of the power, P, is lost to useful effect by friction when A B = 8 times C B.

When the wedge is moved 3 inches the point *q* is raised $\frac{3}{8}$ of an inch, and $m\ n$ $\frac{3}{16}$ of an inch $= \frac{1}{64}$ feet.

It will be shown hereafter that the hammer of the given weight and velocity in twenty blows may develop **13896** units of work to drive the wedge 3 inches, or in raising the centre of gravity $\frac{1}{64}$ part of a foot.

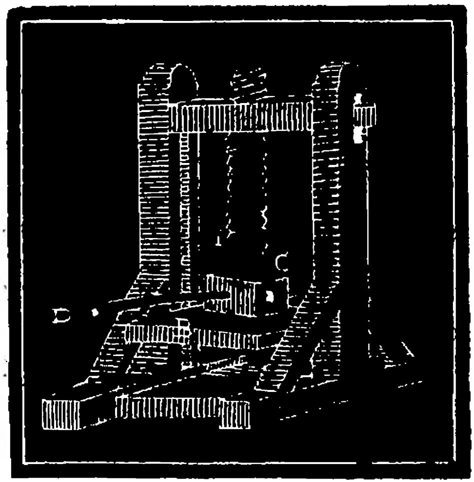
$$\therefore \quad \mathbf{W \times \frac{1}{64} = 13896.}$$

$$\therefore \quad \text{W } 889346 \text{ lbs.} = 397\tfrac{1}{35} \text{ tons.}$$

Hence the friction in this case amounts to a pressure of $97\frac{1}{35}$ tons. (See how to estimate the units of work in a rotating body.)

OF THE SCREW.

Fig. 28.



In this simple machine, Fig. 28, the pressure applied moves in a circle whose radius is the length of the lever C D, or arm of the screw, whilst the direction in which the work is done is a right line.

Ques. 29. The lever C D of a simple screw is 6 feet, the thickness of the thread I=·04 feet; if a pressure, P, of 200 lbs. be applied to the lever, what pressure, W, will be produced on the press board B?

Space moved over by P, in one revolution,

$$=2 \times 6 \times 3{\cdot}1416.$$

Space moved over by W, in one revolution$={\cdot}04$ feet.

Work of P, in one revolution,

$$=\mathbf{2 \times 6 \times 3{\cdot}1416 \times 200};$$

Work of W, in one revolution,

$$=\mathbf{W \times {\cdot}04}.$$

$$\therefore \quad \mathbf{W \times {\cdot}04 = 2 \times 6 \times 3{\cdot}1416 \times 200}.$$

$$\therefore \quad W = 188496 \text{ lbs.}$$

The last question shows that the efficacy of the screw is obtained by increasing the length of the lever, or by decreasing the thickness of the threads.

Ques. 30. The lever, C D, of a screw is 4 feet, and the thickness of the threads $\frac{1}{4}$ inch; required the pressure, P, that must be exerted on the lever to produce a pressure of 12 tons upon the press-board B, Fig. 28?

In one revolution of the lever, the work done,

$$=\mathbf{12 \times 2240 \times \tfrac{1}{48} = 560};$$

for $\frac{1}{4}$ of an inch$=\frac{1}{48}$ of a foot.

In one revolution of the lever the work applied,

$$=P\times 8\times 3{\cdot}1416;$$

$$\therefore\quad P\times 8\times 3{\cdot}1416=560,$$

$$\therefore\quad P=22{\cdot}28 \text{ lbs.}$$

Ques. 31. The lever of a simple screw is 2 feet, what must be the thickness of the threads, so that a pressure of 5000 lbs. may be produced on the press-boards, by a pressure of 100 lbs. on the handle, or lever?

Let x be the thickness of the thread, then, the work applied in one revolution,

$$=4\times 3{\cdot}1416\times 100=1256{\cdot}64;$$

Work done,

$$=x\times 5000=1256{\cdot}64;$$

$$\therefore\quad x={\cdot}2513 \text{ feet.}$$

OF THE COMPOUND SCREW.

THIS mechanical contrivance consists of two screws; one of which screws within the other, so that whilst the large one is descending, the small one is relatively rising within the large one. In consequence of this compound motion, one revolution of the lever causes the press-board only to descend a distance equal to the difference of the thickness of the threads of the screws.

Ques. 32. In a compound screw, the length of the lever is 2·5 feet, the distance between the threads of the large or hollow screw is $\frac{2}{3}$ of an inch, and of the small one half an inch; if 60 lbs. pressure be applied to the lever, what is the pressure on the press-board?

In one revolution the large screw descends $\frac{2}{3}$ of an inch, but, at the same time, the small screw, by turning within the large one, ascends $\frac{1}{2}$ an inch; therefore the press-board must descend a space=

$$\tfrac{2}{3}-\tfrac{1}{2}=\tfrac{1}{6} \text{ of an inch}=\tfrac{1}{72} \text{ of a foot.}$$

Work done in one revolution,

$$=\mathbf{W}\times\tfrac{1}{72};$$

Work applied in one revolution,

$$=\mathbf{60}\times\mathbf{2}\times\mathbf{2\cdot5}\times\mathbf{3\cdot1416}=\mathbf{942\cdot48};$$

$$\therefore\quad \mathbf{W}\times\tfrac{1}{72}=\mathbf{942\cdot48}.$$

$$W=67858\cdot56 \text{ lbs.}$$

Ques. 33. If the length of the lever $=3\frac{1}{2}$ feet, the thickness of the thread of the larger screw $=\frac{3}{4}$ of an inch, what must be the thickness of the thread of the smaller screw, so that 20 lbs. applied to the lever may produce a pressure of 20 tons = 44800 lbs.?

$$\tfrac{3}{4} \text{ of an inch}=\tfrac{1}{16} \text{ of a foot.}$$

Let x be the thickness of the smaller screw in feet.

$$\therefore\quad \mathbf{20}\times\mathbf{7}\times\mathbf{3\cdot1416}=\mathbf{44800}\ (\tfrac{1}{16}-x.)$$

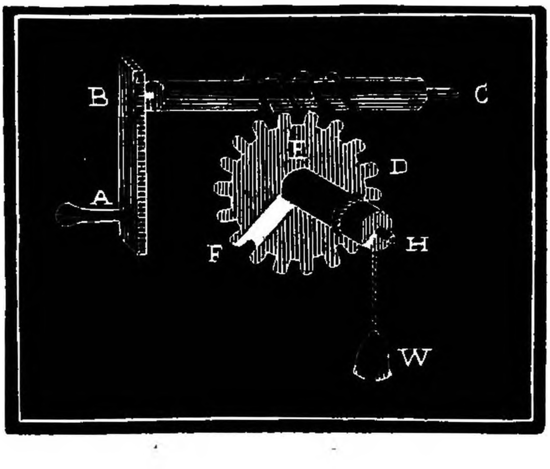
$$\therefore\quad \tfrac{1}{16}-x=\cdot0098175;$$

$$\therefore\quad \cdot0526825=x.$$

THE ENDLESS SCREW.

In this machine, Fig. 29, the threads of a screw cut upon a cylinder, B C, are made to act upon the teeth of a

Fig. 29.



cog-wheel, F D, having an axle, H E, round which a cord coils, as in the common windlass.

This combination gives a very slow motion to the weight, W.

Ques. 34. In an endless screw, the length of the handle, A B, is 3 feet, the number of teeth in the cog-wheel, F D, is 24, and the radius of the axle, E H, $\frac{3}{4}$ of a foot: if 36 lbs. pressure be applied to the handle, what weight, W, will be raised, friction being neglected?

Since the screw is fixed upon the axis of the handle, one turn will cause one of the teeth of the wheel to be moved round; consequently the handle must make 24 turns, whilst the cog-wheel and axle make one turn.

Work done in one revolution of the axle,

$$=W\times\tfrac{3}{4}\times2\times3\cdot1416;$$

for W will be raised 4·7124 feet.

The work applied in one revolution of the axle,

$$=36\times6\times3\cdot1416\times24;$$

$$\therefore\ W\times\tfrac{3}{4}\times2\times3\cdot1416=36\times6\times3\cdot1416\times24;$$

$$\therefore\quad W=\frac{36\times6\times24}{2\times\frac{3}{4}}$$

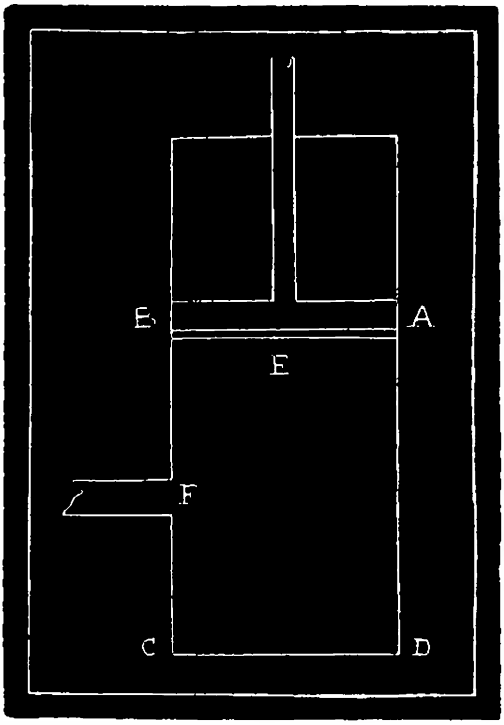
$$W=3456\text{ lbs.}$$

THE HYDROSTATIC PRESS.

If there be any number of pistons of different magnitudes, anyhow applied to apertures in a cylindrical vessel filled with an incompressible and non-elastic fluid, the forces acting on the pistons to maintain an equilibrium will be to one another as the areas of the respective apertures, or the squares of the diameters of the pistons.

Let A B C D, Fig. 30, represent a section passing along the axis of a cylindrical vessel filled with an incompressible

Fig. 30.



and non-elastic fluid, and let E F be two pistons of different magnitudes, connected with the cylinder and closely fitted to their respective apertures or orifices; the piston F being applied to the aperture in the side of the vessel, and the piston E occupying an entire section of the cylinder or vessel, by which the fluid is contained.

Then, because by the nature of fluidity, the pressures on every part of the pistons E and F are mutually transmitted to each other through the medium of the intervening fluid, it follows that these pressures will be in a state of equilibrium when they are equal among themselves.

The pressure of the power is applied to the small piston by a lever of the second kind, and the advantage of the machine depends upon the length of this lever, and the extent of surface of the large piston, compared with that of the small one.

Ques. 35. In a hydrostatic press, the surface of the pistons F, E, Fig. 30, are 3 and 350 square inches respectively, the lever is 30 inches long, and the piston-rod is attached 2 inches from the fulcrum: if a pressure of 100 lbs. be applied to the lever, what pressure will be produced upon the large piston?

Let the small piston descend $\frac{1}{3}$ of an inch, or $\frac{3}{16}$ part of a foot, then 1 cubic inch of water will be thrown into the large cylinder, and its piston will be raised $\frac{1}{350}$ part of an inch, or $\frac{1}{4200}$ of a foot;

Work on the small piston,

$$=\frac{100\times 30}{2}\times\frac{1}{36}=41\tfrac{2}{3}$$

Work on the large piston,

$$=W\times\frac{1}{4200};$$

$$\therefore\quad \frac{W}{4200}=41\tfrac{2}{3}$$

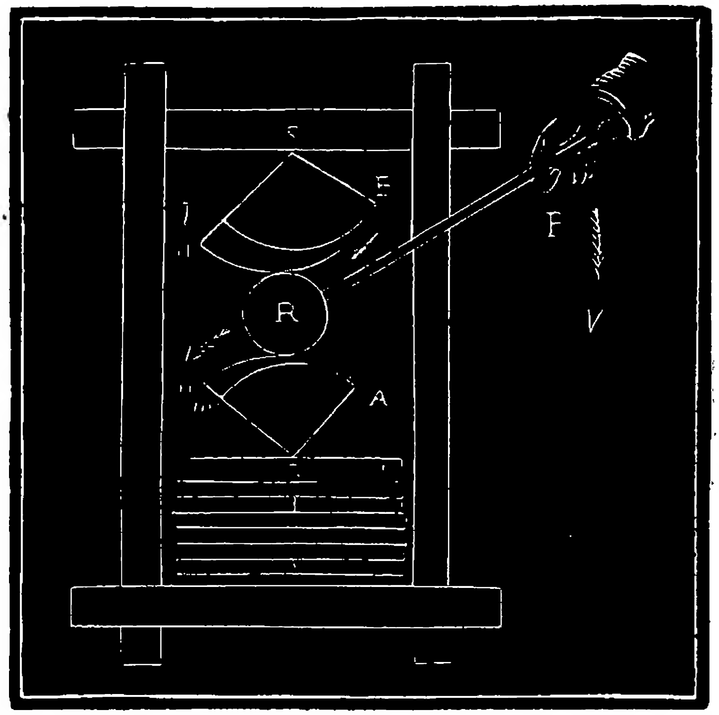
$$W=175000 \text{ lbs.}$$

DICK'S ANTI-FRICTION CAM-PRESS.

THIS powerful machine, invented by David Dick, being almost free from friction and capable of so many useful modifications, may be ranked among the most ingenious contrivances of modern times. It obtained one of the highest prizes at the Paris International Exhibition. In the anti-friction cam-press, Fig. 31, let the power P turn the roller R, and also the cams A Q n, B S a; suppose the circumference of the roller R = the length of the arc n A = B a; the points S R Q retained in the same straight line; the point S stationary, but R and Q movable in the direction of the arrow Q. Then it is evident, when P makes one revolution, the press-board, C D, will move through a space equal to the difference of $a\,b$ and $m\,n$,

$$Q\,m = Q\,A,$$
$$S\,b = S\,B.$$

Fig. 31.



But by turning one of the cams, as A Q $m\,n$, so that the side Q $m\,n$ take the position of the side Q A, then C D will move through a space $= a\,b + m\,n$.

Ques. 36. Let a pressure of 210 lbs. = P, perpendicular to R P, describe a circular path = 24 feet; Q A = Q m = b S = S B; $a\,b = 2$ inches, $m\,n = 2{\cdot}15$ inches, and the circumference of R = 9 inches = the length of the arc aB = nA; required the pressure Q on the press-board C D?

The units of work developed by P in one revolution equal

$$24 \times 210 = 5040 =$$

the units of work expended on the surface of R, or on the inclined plane whose length $= a\,b = n\,\mathrm{A} =$ the circumference of R = 9 inches $= \frac{3}{4}$ feet. The height of this inclined plane = the difference of $a\,b$, and $m\,n = {\cdot}15$ inches = ·0125 feet;

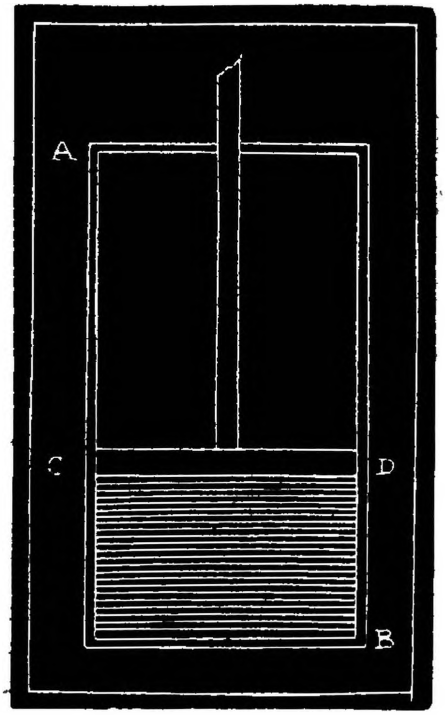
$$\therefore \quad Q \times {\cdot}0125 = 5040.$$

$$\therefore \quad Q = \frac{5040}{{\cdot}0125} = 403200 \text{ lbs.} = 180 \text{ tons.}$$

CHAPTER V.

WORK OF STEAM AND THE STEAM ENGINE.

Fig. 32.



Let A B be a steam cylinder, Fig. 32, and suppose the steam to exert a mean or constant pressure upon the piston C D; if this constant pressure be 35 lbs. to the square inch, then if a weight or pressure of 35 lbs. be placed upon every square inch of the piston, the steam would just be capable to move the length of the stroke. Therefore the units of work performed upon each square inch of the piston in one stroke will be found by multiplying the pressure of the steam upon 1 inch by the length of the stroke in feet, and the units of work upon the whole piston will be the work upon 1 inch multiplied by the number of square inches of the whole piston. In high-pressure engines the pressure of the atmosphere is opposed to the pressure of the steam. Besides the pressure of the atmosphere, which is about 14·7 lbs. to 15 lbs. to the square inch, a pressure of about 1 lb. to the square inch is taken for the friction due to the engine when unloaded, with an additional friction of about $\frac{1}{7}$ the useful load or effective pressure, to allow for the resistance necessary to overcome the friction of the loaded engine. For example, if the pressure of the steam = 56 lbs. as it comes from the boiler, from which if 15 lbs. be taken for the pressure of the atmosphere, and 1 lb. for the friction of the unloaded engine, then the remaining 40 lbs. to the square inch will be taken up by the *useful load* and its friction.

$$\therefore \quad \text{load} + \frac{\text{load}}{7} = 40 \text{ lbs.}$$

$$\text{or} \quad \frac{8 \text{ load}}{7} = 40 \text{ lbs.}$$

∴ in this case, the useful load, or effective pressure of the steam = 35 lbs.

When a condensing engine is employed, the pressure in the condenser is deducted instead of that of the atmosphere; it is seldom more than 3½ or 4 lbs. to the square inch.

∴ For a condensing engine

$\frac{8}{7}$ of load+1+4=total or mean pressure of the steam.

In a condensing engine let the total pressure of the steam = 61 lbs. to the square inch.

$$\therefore \quad \tfrac{8}{7} \text{ load} = 61 - 5 = 56\,;$$

$$\therefore \quad \text{load} = 49.$$

that is, the effective pressure of the steam in this case=49 lbs.

The allowances given above in round numbers may be modified to suit particular cases without effecting the principle of work upon which our calculations are based.

EXAMPLES.

Question 1. The area of the piston of a steam engine=2200 square inches, the mean effective pressure of the steam 15 lb. the square inch, the length of the stroke 8 feet, the number of strokes a minute=23; what is the horse power of the engine?

Work done upon 1 square inch of the piston in one stroke,

=15×8=120.

Work upon the whole piston in one stroke,

=120×2200=264000

Work done in one minute, or in 20 strokes,

=264000×20=5280000.

But the power of a horse being,

$$=33000'.$$

$$\therefore \frac{5280000'}{33000'}=160^{\text{h}}.$$

Ques. 2. The area of the piston of a high-pressure engine is 625 square inches, the length of the stroke 6 feet, the pressure of the steam 40 lbs. the square inch, the number of strokes a minute=20; it is required to find the number of cubic feet of water which the engine will pump from a mine whose depth is 396 feet, making the usual allowance for friction and the modulus of the pump.

The *modulus* of a machine is the fraction which expresses the relation of the work done to the work applied. For example, if a machine should only perform one-half the work that is applied to it, then the *modulus* in this case would be $\frac{1}{2} = \cdot5$. However perfectly a machine may be constructed, there must always be a certain amount of work destroyed by friction. The following table of machines for the raising of water is taken from *Morin's Méchanique Pratique*.

	Modulus
Incline Chain pump	·38
Upright Chain pump	·53
Bucket Wheel	·60
Chinese Wheel	·58
Archimedian Screw	·70
Pumps for draining mines	·66

With respect to the present question we take the *modulus* ·66. Then,

$$\text{Load}+\tfrac{1}{7}\text{ load}+1+15=40;$$

$$\frac{8\text{ load}}{7}=24;$$

$$\text{load}=21\text{ lbs.}$$

Useful work of the engine a minute,

$$=21\times625\times6\times20\times\cdot66=1039500'.$$

Work to pump one cubic foot of water a height of 396 feet,

$$=62\cdot5\times396=247500,$$

Consequently the number of cubic feet that will be pumped a minute,

$$=\frac{1039500'}{24750}=42 \text{ cubic feet.}$$

Ques. 3. The area of the piston of a high-pressure engine = 660 square inches, the length of stroke = 6 feet, the number of strokes a minute = 20; what must be the mean effective pressure of the steam, so that the engine may do the work of 25 horses?

Let x be the number of lbs. pressure on each square inch of the piston, then, the work a minute,

$$=x\times 660\times 6\times 20;$$

$$\therefore\quad x\times 660\times 6\times 20=25\times 33000;$$

$$\therefore\quad x=\frac{825000}{79200}=10\tfrac{5}{12} \text{ lbs.}$$

$$\begin{array}{rr} 7) & 10\cdot4166 \\ & 1\cdot4880 \\ \hline & 11\cdot9046 \\ & 15\cdot0000 \\ & 1\cdot0000 \\ \hline & 27\cdot9046 \end{array} \text{ lbs. total pressure of the steam.}$$

Ques. 4. The area of a piston of a high-pressure engine = 3000 square inches, the length of stroke = 11 feet, the number of strokes a minute = 16; required the mean pressure of the steam, so that the engine may perform the work of 224 horses, making the usual allowance for friction.

Let x lbs. be the effective pressure of the steam.

Work per minute with x lbs. effective pressure,

$$=x\times 3000\times 11\times 16=528000\times x;$$

The effective work per minute will also be expressed by

$$33000\times 224=7392000'.$$

$$\therefore\quad 528000\times x=7392000'.$$

$$\therefore\quad x=14 \text{ lbs. effective pressure.}$$

$$1+15+14+\tfrac{1}{7} \text{ of } 14=32 \text{ lbs. mean pressure of steam.}$$

Ques. 5. The length of the stroke of a high-pressure engine = 8 feet, the area of the piston 1000 inches, and the number of strokes a minute = 20; what must be the pressure of the steam so that the engine may pump 66 cubic feet of water a minute from a mine whose depth is 896 feet, making the usual allowances for friction and modulus of the pump?

The work done in one minute,

$$=66\times 62{\cdot}5\times 896=3696000';$$

The modulus of the pump $={\cdot}66$;

Hence the work of the engine a minute $\times{\cdot}66$

$=3696000'$, therefore the work a minute

$$=\frac{3696000'}{{\cdot}66}=5600000'.$$

The useful work done one inch of the piston in one stroke,

$$=\frac{5600000'}{1000\times 20}=280.$$

$$\text{Useful load}=\frac{280}{8}=35 \text{ lbs.};$$

consequently the pressure of the steam

$$=35+\frac{35}{7}+15+1=56 \text{ lbs.}$$

In the steam engine, the source of work is the evaporating power of the boiler.

The magnitude of the work depends upon the quantity of water evaporated in a given time, and also upon the temperature, and consequently the pressure at which the steam is formed. Experimental tables have been framed, giving the relation of the volume and pressure of steam from a cubic foot of water; *from these* tables may be found the volume of steam when its pressure and volume of water are given, and *vice versâ.* The following will serve as a specimen of such tables:

VOLUME OF A CUBIC FOOT OF WATER IN THE FORM OF STEAM AT THE CORRESPONDING PRESSURES AND TEMPERATURES.

Total pressure in pounds the square inch.	Corresponding temperature, by Fahrenheit's thermometer.	Volume of the Steam compared with the volume of the water that has produced it.	Total pressure in pounds the square inch.	Corresponding temperature, by Fahrenheit's thermometer.	Volume of Steam; volume of water=1.
1	109·2	20954	37	263·7	727
2	126·1	10907	38	265·3	710
3	141·0	7455	39	266·9	693
4	152·3	5695	40	268.4	677
5	161·4	4624	41	269·9	662
6	169·2	3901	42	271·4	647
7	176·0	3380	43	272·9	634
8	182·0	2985	44	274·3	620
9	187·4	2676	45	275·7	608
10	192·4	2427	46	277·1	596
11	197·0	2222	47	278.4	584
12	201·3	2050	48	279·7	573
13	205·3	1903	49	281·0	562
14	209·0	1777	50	282·3	552
15	213·0	1669	51	283·6	542
16	216·4	1572	52	284·8	532
17	219·6	1487	53	286·0	523
18	222·6	1410	54	287·2	514
19	225.6	1342	55	288·4	506
20	228·3	1280	56	289·6	498
21	231·0	1224	57	290·7	490
22	233·6	1172	58	291·9	482
23	236·1	1125	59	293·0	474
24	238·4	1082	60	294·1	467
25	240·7	1042	61	294·9	460
26	243·0	1005	62	295.9	453
27	245·1	971	63	297·0	447
28	247·2	939	64	298·1	440
29	249·2	909	65	299·1	434
30	251·2	882	66	300·1	428
31	253·1	855	67	301·2	422
32	255·0	831	68	302·2	417
33	256·8	808	69	303·2	411
34	258·6	786	70	304·2	406
35	260·3	765	71	305·1	401
36	262·0	746	72	306·1	396

Total pressure in pounds the square inch.	Corresponding temperature, by Fahrenheit's thermometer.	Volume of the Steam compared with the volume of the water that has produced it.	Total pressure in pounds the square inch.	Corresponding temperature, by Fahrenheit's thermometer.	Volume of Steam; volume of water = 1.
73	307·1	391	87	319·4	333
74	308·0	386	88	320·3	330
75	308·9	381	89	321·1	326
76	309·9	377	90	321·9	323
77	310·8	372	91	322·7	320
78	311·7	368	92	323·5	317
79	312·6	364	93	324·3	313
80	313·5	359	94	325·0	310
81	314·3	355	95	325·8	307
82	315·2	351	96	326·6	305
83	316·1	348	97	327·3	302
84	316·9	344	98	328·1	299
85	317·8	340	99	328·8	296
86	318·6	337	100	329·6	293

Ques. 6. In a high-pressure engine, the area of the piston =120 square inches, the length of the stroke=2·4 feet, the effective evaporation of the boiler=·5 cubic feet a minute, the pressure of the steam in the cylinder 64 lbs. on a square inch; making the usual allowance for the loss due to friction, required the useful load on each square inch of piston, and the useful horse power?

$$\tfrac{8}{7}\text{ useful load}=64-15-1=48.$$
$$\text{Useful load}=42\text{ lbs.}$$

From the table it will be found that a cubic foot of water raised to steam of 64 lbs. pressure has a volume of 440 cubic feet.

Volume of steam evaporated a minute

$$=440\times{\cdot}5=220\text{ cubic feet;}$$

Volume discharged at each stroke

$$=\frac{120\times 2{\cdot}4}{144}=2\text{ cubic feet;}$$

The number of strokes each minute

$$=\frac{220}{2}=110.$$

The work in one stroke

$$=42\times120\times2{\cdot}4=12096.$$

Work in one minute

$$=12096\times110=1330560'.$$

Horse power $=\frac{1330560'}{33000'}=40^{h}\cdot32.$

Ques. 7. The area of the piston of a high-pressure engine is 264 square inches, the length of stroke $5\frac{1}{2}$ feet, the pressure of the steam in the cylinder=72 lbs. the square inch, the number of strokes a minute=36; required the useful load, the water evaporated each hour, and the useful horse power of the engine?

Let x be the useful load, then

$$x+\frac{x}{7}+15+1=72\text{ lbs.}$$

$$7x+x=392.$$

$$x=49\text{ lbs.}$$

The volume of steam discharged a minute, in cubic feet

$$=\frac{264\times5{\cdot}5}{144}\times36=363\text{ cubic feet.}$$

The cubic feet of steam discharged in an hour

$$=363\times60=21780.$$

From the table it will be found that one cubic foot of water yields 396 cubic feet of steam at 72 lbs. pressure.

Consequently, the cubic feet of water evaporated each hour

$$=\frac{21680}{396}=55.$$

The useful horse power of the engine

$$=\frac{264\times49\times5{\cdot}5\times36}{33000'}=77^{h}\cdot616.$$

It has been found by experiment, that whatever may be the pressure at which steam is formed, the quantity of fuel necessary to evaporate a given volume of water is nearly the same. Hence it follows, that it is most advantageous to employ steam of a high pressure.

Ques. 8. Required the *duty* of the engine, that is, the units of work developed by a bushel of coal each hour; in the last example, allowing a bushel of coals 94 lbs. to evaporate 11·5 cubic feet of water?

The useful work per hour

$$=264\times49\times5\tfrac{1}{2}\times36\times60=153679680;$$

which is the work of 55 cubic feet of water. Therefore the work of a bushel of coals, or 11·5 cubic feet of water,

$$=153679680\times\frac{11\cdot5}{55}=32133024,$$

the duty of the engine.

Ques. 9. A train of 200 tons moves along at a uniform speed of 30 miles an hour upon a level rail, the resistance of friction upon the rail is $5\frac{1}{2}$ lbs. a ton, the resistance of the atmosphere 33 lbs. upon the whole train when the speed is 10 miles an hour, the diameter of the driving-wheel 7 feet, the area of the piston 120 square inches, the length of the stroke$=\frac{3}{2}$ feet, and in addition to the resistances of friction, the resistance due to the blast pipe is $1\frac{3}{4}$ lbs. on the square inch of the piston, when the speed of the train is 10 miles an hour. Required the pressure of the steam, the evaporation of the boiler, and the number of bushels of coal necessary for a journey of 500 miles, supposing that 1 bushel, or 94 lbs., will evaporate $11\frac{1}{2}$ cubic feet of water?

When the diameter of a circle$=7$ feet, the circumference$=22$ feet nearly, which is the space passed over in one revolution of the driving-wheel.

Total resistance to the motion of the carriages

$$=200\times5\cdot5+\left(\frac{30}{10}\right)^2\times33=1397\text{ lbs.}$$

The work in one revolution

$$=1397\times22=30734.$$

On the supposition that 22 is the circumference when 7 = the diameter.

Work of 1 lb. pressure on one square inch of the piston in one revolution

$$=1\times120\times\frac{3}{2}\times4=720,$$

for the engine has two cylinders, and each piston makes two strokes while the driving-wheel turns once round.

Consequently, the effective pressure on one square inch of the piston,

$$=\frac{30734}{720}=42\cdot68\text{ lbs.}$$

It has been found by experiment, that the resistance of the blast pipe increases with the speed of the engine,

∴ Resistance due to the blast pipe

$$=1\frac{3}{4}\times\frac{30}{10}=4\tfrac{1}{4}\text{ lbs.}$$

Hence the total pressure of the steam on the piston

$$=42\cdot68+\frac{42\cdot68}{7}\times1+15+4\tfrac{1}{4}=69\cdot03\text{ lbs.}$$

The number of revolutions of the driving wheel a minute

$$=\frac{30\times5280}{60\times22}=120;$$

therefore, the two pistons will make 130 × 4 = 480 strokes a minute.

The cubic feet of steam discharged a minute will be

$$=\frac{120}{144}\times\frac{3}{2}\times480=600.$$

From the table it will be found that a cubic foot of water produces 411 cubic feet of steam of 69·03 lbs. pressure.

The number of cubic feet of water evaporated a minute

$$=\frac{600}{411}.$$

As one bushel of coal evaporates 11·5 cubic feet of water, the number of bushels of coal a minute

$$=\frac{600}{411\times 11\cdot 5};$$

A journey of 500 miles is performed in 16⅔ hours, or in 1000 minutes, at the rate of 30 miles an hour.

Therefore the number of bushels of coal for 1000 minutes

$$=\frac{600\times 1000}{411\times 11\cdot 5}=126\cdot 95 \text{ bushels.}$$

Ques. 10. In a locomotive engine, the area of the piston is 90 inches, the length of the stroke 16 inches, the pressure of the steam 50 lbs., the effective evaporation of the boiler ·7 cubic feet a minute, the diameter of the driving-wheel 5 feet; what is the speed of the train an hour?

At 50 lbs. pressure, 1 foot of water forms 554 cubic feet of steam.

The volume of steam generated a minute

$$=554\times \cdot 7=387\cdot 8.$$

Cubic feet discharged in one revolution of the driving-wheel

$$=\frac{4\times 90\times 16}{1728}=3\tfrac{1}{3}.$$

Therefore the number of revolutions of the driving-wheel a minute

$$=\frac{387\cdot 8}{3\cdot 33}=116\cdot 3.$$

∴ The space moved over by the carriage a minute in feet

$$=5\times 3\cdot 1416\times 116\cdot 3.$$

Consequently, the miles moved over an hour

$$=\frac{5\times 3\cdot 1416\times 116\cdot 3\times 60}{5280}=20\cdot 75955.$$

Ques. 11. The area of the piston of a locomotive engine =80 square inches, the length of the stroke=15 inches, the pressure of the steam 48 lbs. the square inch, and the diameter of the driving-wheel 5 feet; required the effective evaporation of the boiler, so that the train may have a speed of 30 miles an hour; the effective horse power of the engine and the weight of the train are also required, taking the resistances to the motion of the piston as in question 9?

Number of revolutions of the wheel a minute

$$=\frac{30\times 5280}{60\times 5\times 3{\cdot}1416}=168.$$

Number of strokes of both pistons a minute

$$=168\times 4=672.$$

$$\text{Load}+\frac{\text{load}}{7}+15+1+\frac{30}{10}\times\ 1{\cdot}75=48\text{ lbs.}$$

$$\text{Load}=\frac{26{\cdot}75\times 7}{8}=23{\cdot}4\text{ lbs.}$$

The effective work a minute will be

$$=\mathbf{23{\cdot}4}\times\mathbf{80}\times\tfrac{5}{4}\times\mathbf{672}=\mathbf{1572480'},$$

Hence, the effective horse power

$$=\frac{\mathbf{1572480'}}{\mathbf{33000'}}=\mathbf{47^{h}\cdot 65}.$$

Now the effective work has to support a speed of 30 miles an hour, 2640 feet a minute, opposed by the resistances of friction and the atmosphere.

Work due to the resistance of the atmosphere a minute

$$=\left(\frac{\mathbf{30}}{\mathbf{10}}\right)^2\times\mathbf{33}\times\mathbf{2640}=\mathbf{784080'}.$$

Therefore the work due to friction a minute

$$=\mathbf{1572480'}-\mathbf{784080'}=\mathbf{788400'}.$$

Work of friction a minute when the train weighs x tons

$$=\mathbf{7}\times\mathbf{2640}\times x;$$

$$\therefore\ x=\frac{\mathbf{788400'}}{\mathbf{7}\times\mathbf{2640}}=42{\cdot}6\text{ tons.}$$

Cubic feet of steam discharged each stroke

$$=\frac{80\times 1{\cdot}25}{144};$$

the cubic feet discharged in a minute

$$=\frac{80\times 1{\cdot}25\times 672}{144},$$

From the table it will be found, that a cubic foot of water yields 573 cubic feet of steam; therefore, the number of cubic feet of water evaporated a minute

$$=\frac{80\times 1{\cdot}25\times 672}{144\times 573}=\cdot 81 \text{ cubic feet.}$$

Ques. 12. A railroad train of 60 tons ascends an incline that has a rise of $\frac{1}{4}$ in 100; required the maximum speed an hour, 40 being the effective horse power of the engine, friction 8 lbs. a ton?

Let $x=$ the speed of the train in feet an hour.

$$100 : \tfrac{1}{4} :: x : \frac{x}{400}=$$

the rise of the rail in x feet.

Work due to gravity in an hour,

$$=\frac{60\times 2240\times x}{400}=336\,x.$$

Work of friction, $=60\times 8\times x=480\,x$;

But the work due to gravity each hour, added to the work due to friction each hour, must be equal to the work done by the engine in the same time;

$$\therefore\ 816\,x=40\times 33000\times 60=79200000.$$

$$\therefore\ x=\frac{79200000}{816}=97059 \text{ ft.}=18{\cdot}4 \text{ miles.}$$

If the engine in the last example move this train, what must be the effective evaporation of the boiler, and the duty of the engine?

Speed a minute, $=\frac{97059}{60}=1617{\cdot}6$ feet.

Number of strokes of the pistons a minute,

$$=\frac{1617{\cdot}6}{5\times 3{\cdot}1416}\times 4=412;$$

The effective work of the engine a minute,

$$=33000\times 40=1320000'.$$

Suppose $y=$ the pounds effective pressure on each square inch of the piston, then the work of y lbs. effective pressure a minute

$$y \times \mathbf{80} \times \tfrac{15}{12} \times \mathbf{412} = \mathbf{1320000}'.$$

$$\therefore \quad y = 32 \text{ lbs.}$$

Pressure of steam

$$= 32 + \tfrac{1}{7} \times 32 + 15 + 1 + \frac{18{\cdot}4}{10} \times 1{\cdot}75 = 55{\cdot}7 \text{ lbs.}$$

One cubic foot of water in the form of steam, at 55·7 lbs. pressure, is 504 cubic feet. Number of cubic feet discharged a minute, in the form of steam,

$$= \frac{80 \times 15}{1728} \times 412 = 287.$$

∴ Number of cubic feet of water evaporated a minute

$$= \frac{287}{504} = {\cdot}57.$$

Now this water performs **1320000′**,

∴ 11·5 cubic feet of water, or 94 lbs. of coal,

$$= \frac{\mathbf{11{\cdot}5}}{\mathbf{{\cdot}57}} \times \mathbf{1320000}' = \mathbf{26631579},$$

the duty of the engine.

Under the authority of an act of the American Congress, approved September 11th, 1841, an extensive series of experiments was conducted by Professor Johnson upon the evaporative power of several kinds of coal. The number of samples tried was 41, including 9 anthracite from Pennsylvania; 6 foreign bituminous coals, namely: 1 from Sydney, Nova Scotia; 2 of Picton coal; 1 Scotch; 1 of Newcastle; 1 of Liverpool. From 1 to 6 trials were made on each sample; the average quantity used per trial being 978 lbs. The experiments occupied 144 days, during each of which continuous observations were made during 12 to 14 hours. The coals were burnt under a steam boiler with apparatus for complete regulation, the supply of water and coals being determined both by weight and measure. The standard adopted to measure the heating power of each kind of coal was the weight of water which a given weight of each evaporated from the temperature 212° Fahr. The following Table gives the results of the comparisons of six qualities, in each of which, that coal which ranks the highest is stated as 1000, and the others in decimal parts of the integer.

KINDS OF COAL.	Pounds of steam raised from water 212°, by 1 lb. of fuel.	Relative evaporative power for equal weights of coal.	Pounds of steam from 212°, produced by one cubic foot of each.	Relative evaporative power for equal bulks of coal.	Per-centage of total waste in clinker and ashes.	Relative freedom from waste.	Time required to bring the boiler to steady action in hours.	Relative rapidities of ignition.	Pounds of unburnt coke on the grate after each trial.	Relative completeness of combustion.	Weight in pounds to a cubic foot by experiment.	Relative weights.
ANTHRACITE COALS.												
Atkinson's and Templeman's...........	10·70	1·000	566·2	1·000	7·96	·633	·99	·505	5·1	·725	52·92	·945
Beavor Meadow Slope (No: 5)......	9·88	·923	556·1	·982	6·74	·748	2·42	·207	61·2	·060	51·19	1·000
BITUMINOUS AND FREE-BURNING COAL.												
Newcastle...............	8·66	·809	439·6	·776	5·68	·887	·84	·595	10·7	·346	50·82	·904
Picton.....................	8·48	·792	417·9	·738	1·206	·418	·85	·588	3·7	1·000	49·25	·876
Liverpool................	7·84	·733	375·4	·663	5·04	1·000	·86	·581	11·1	·333	47·88	·852
Cannelton, Indiana...	7·34	·686	348·8	·616	5·12	·984	·50	1·000	6·4	·578	47·65	·846
Scotch......................	6·95	·649	353·8	·625	1·010	·499	·96	·521	5·7	·649	51·05	·909
Dry pine-wood	4·69	·436	98·6	·175	·307	16·417						

UNITS OF WORK DEVELOPED BY CONDENSING STEAM.

EXPERIMENTS show that when water is raised to the temperature of 216·3° Fahr., or 4·3° above the boiling point, the volume is increased 1573 times, and the pressure will be 16 lbs. on the square inch. Suppose steam of this temperature to enter the lower part of the cylinder A C, Fig. 33, the piston C will rise, as 14·7 lbs. may be taken as the mean pressure of the atmosphere. Then let the steam be condensed by cold water; a vacuum will be formed, and the piston pressed down by the whole weight of the atmosphere on each square inch of the piston, if the vacuum be complete; but it has been found that a perfect vacuum cannot be formed in this way, as water gives off vapour at all temperatures. If the temperature of the vapour in condenser be 103° Fahr., the pressure will be 1 lb. to the square inch:

For	130°	pressure	2·129	lbs.
„	145°	„	3·10	„
„	150°	„	3·64	„
„	165°	„	5·23	„
„	170°	„	5·94	„
	&c.		&c.	

EXAMPLES.

Ques. 13. Required the work developed by expanding and condensing a cubic foot of water, supposing 4 lbs. to be the pressure of the vapour after condensation, and 15 lbs. on the square inch to be the pressure of the steam; find, also, the duty of the atmospheric engine using the steam in this manner.

Cubic feet of vacuum formed by condensation
=1710=1711−1.

Pressure on one square inch of the piston
=14·7−4=10·7 lbs.

Suppose the area of the piston to be one square foot, the length of the stroke will be 1710 feet.

Consequently, the work of one cubic foot of water

=144×10·7×1710=2634768.

And the duty or work of 94 lbs. of coal=1 bushel, =duty of $11\frac{1}{2}$ cubic feet of water

$$=2634768 \times 11 \cdot 5 = 30299832.$$

Ques. 14. What must be the effective evaporation of the boiler of an atmospheric engine, so that the horse power may be 30, allowing 3 lbs. for the elasticity of the vapour in the condenser; the volume of steam being 1700 times the volume of the water from which it is produced?

Work of the engine a minute

$$30 \times 33000' = 990000'.$$

Work of one cubic foot of water

$$=(15-3) \times 144 \times 1700 = 2937600.$$

Therefore, the number of cubic feet of water evaporated a minute

$$=\frac{990000'}{2937600} = \cdot 34.$$

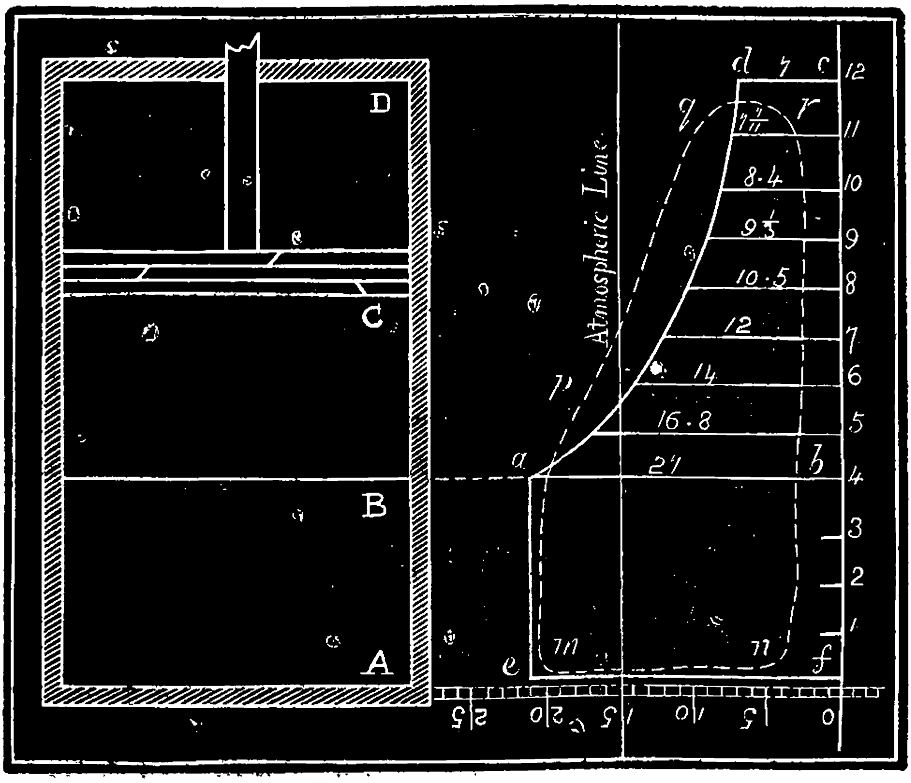
It has been found by experiment that 1 lb. of good Newcastle coal will evaporate $8\frac{2}{3}$ lbs. of water in an hour; therefore, a bushel of this coal will evaporate $814\frac{2}{3}$ lbs. of water. Then, if a cubic foot of water = $62\frac{1}{2}$ lbs., a bushel of coal will evaporate 13 cubic feet of water an hour. Hence the duty of an engine varies with the nature of the water and coal employed.

OF THE USE OF STEAM EMPLOYED EXPANSIVELY.

If steam enter the cylinder for only a part of the stroke, and then, for the remaining portion, the piston is moved by allowing the steam to expand, it is said to be used expansively. This is the most economical way of employing steam; for all the available work may be taken out of the elastic vapour before it is condensed.

When the volume of steam is increased, its elasticity or pressure is decreased in the same ratio; that is, if its volume is increased three times, its pressure will be one-third of what it was at first, and so on. This is Boyle's law: but M. Regnault has shown that it will not hold in extreme cases.

Fig. 33.


D
C
B
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7
c
q
r
$7\frac{7}{11}$
8·4
$9\frac{1}{3}$
10·5
12
14
16·8
21
Atmospheric Line
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3
2
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25
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0

TO FIND THE POWER AND ELASTIC FORCE OF STEAM IN UNITS OF WORK.

When the student does not understand the simple process of calculating dual logarithms, Thomas Simpson's rule may be applied to obtain approximate results.

RULE.

Divide that part of the stroke through which the expansion takes place into any even number of equal parts, and calculate the pressure on the square inch upon the piston at each division of the stroke; take the sum of the extreme pressure in pounds on the square inch, four times the sum of the even pressures, and twice the sum of the odd pressures; multiply the sum of all these by one-third of the common distance between the positions of the piston, and the result will be the work done upon each square inch of the piston after expansion begins. The work done before expansion begins is evidently equal to the pressure on the square inch, multiplied by the number of feet described before expansion. The whole work done during the stroke is equal to the sum of the works done before and after expansion.

EXAMPLE.

Ques. 15. The pressure of steam upon the piston, Fig. 33, is 21 lbs. on the square inch; the length of the stroke, 12 feet; the steam cut off at 4 feet: find the number of units of work done upon each square inch of the piston; supposing the resistance arising from imperfect condensation 2½ lbs. on the square inch.

BY BOYLE'S LAW.

Pressure on the piston from A to B, or for 4 feet of the stroke=21 lbs. The steam being cut off at 4 feet, divide the remaining part of the stroke, namely, 8 feet, into 8 equal parts. To find the pressure at the end of 5 feet,

5 : 4 : : 21 : 16·8 lbs.

To find the pressure at the end of 6 feet,

6 : 4 : : 21 : 14 lbs.

To find the pressure at the end of 7 feet,

7 : 4 : : 21 : 12 lbs.

The remaining pressures are found in the same manner, and are marked on the diagram (see Fig. 33).

16·8		
12·0	14·0	
9·3	10·5	21
7·6	8·4	7
45·7	32·9	28
4	2	
182·8	65·8	

Four times the even ordinates = 182·8
Twice the odd ordinates . . = 65·8
Sum of extreme ordinates . = 28·0

3) 276·6

Area of *a b c d* = **92·2**

Area of *a e f b* = 21 × 4 = **84·0**

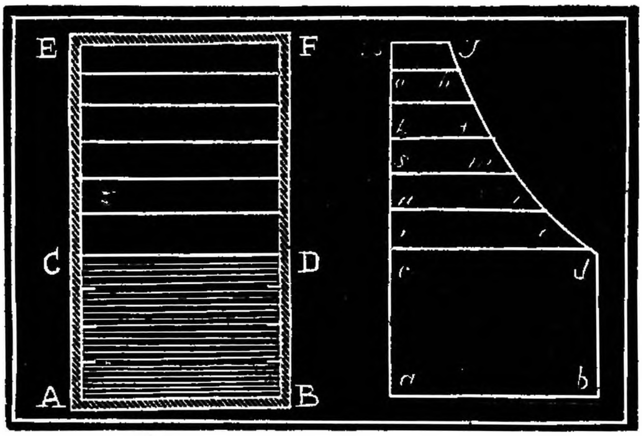
Area of *f e a d c* = **176·2**

Resistance of uncondensed vapour

$$= 2\tfrac{1}{2} \times 12 = 30.$$

176·2 − 30 = 146·2 units of work done on each square inch of the piston in one stroke.

Fig. 34.



A B represents a steam cylinder, the steam cut off at C D; *a x y d b* is a plane figure, *a c*=2; *a o*=$2\frac{1}{2}$; *a n*=3; *a s*=$3\frac{1}{2}$; &c., taken from any scale of equal parts. *c d*=48; *o e*=38·4; *n r*=32 &c., taken from any other scale of equal parts.

EXAMPLES.

Ques. 16. In a condensing engine the length of the stroke is 5 feet, the steam is cut off at 2 feet of the stroke, the pressure of the steam in the cylinder 48 lbs., and the elasticity of the vapour in the condenser is 4 lbs.; required the work performed upon one square inch of the piston in one stroke?

Let the space through which the steam acts expansively be divided into six equal parts, then each interval will be $\frac{1}{2}$ a foot; then by Boyle's law:

$$c\,d = 48 = 48\cdot \text{ lbs.}$$

$$o\,e = \frac{2 \times 48}{2\frac{1}{2}} = 38\cdot4 \text{ lbs.}$$

$$n\,r = \frac{2 \times 48}{3} = 32\cdot \text{ lbs.}$$

$$s\,m = \frac{2 \times 48}{3\frac{1}{2}} = 27\cdot4 \text{ lbs.}$$

$$k\,f = \frac{2 \times 48}{4} = 24\cdot \text{ lbs.}$$

$$g\,h = \frac{2 \times 48}{4\frac{1}{2}} = 21\cdot3 \text{ lbs.}$$

$$x\,y = \frac{2 \times 48}{5} = 19\cdot2 \text{ lbs.}$$

Area of $c\,d\,y\,x$, or work done expansively on 1 square inch of the piston in one stroke, according to Simpson's rule,

$$= \frac{\frac{1}{2}}{3}\left\{48 + 19\cdot2 + 4(38\cdot4 + 27\cdot4 + 21\cdot3) + 2(32 + 24)\right\} = 87\cdot9;$$

The work done before the steam is cut off, that is, from A B to C D,

$$48 \times 2 = 96;$$

Work done against the piston by the vapour in the condenser

$$= 4 \times 5 = 20.$$

Consequently, the total work on one inch of the piston in one stroke

$$= 87\cdot9 + 96 - 20 = 163\cdot9.$$

Ques. 17. The area of a piston of a condensing engine is 1320 square inches, the length of the stroke, including clearance, $= 7$ feet; the steam is cut off at 3 feet of cylinder, the clearance $= \frac{1}{6}$ feet, the pressure of the steam $= 25{\cdot}2$ lbs., the elasticity of the vapour in the condenser $= 3{\cdot}5$ lbs., the effective evaporation of the boiler ·18 cubic feet a minute, the resistances are roughly approximated to as in former questions; required the useful load, and the useful horse power of the engine?

By dividing the space through which the steam acts expansively into four equal parts, and calculating the pressure by Boyle's laws: first $= 25{\cdot}2$, second $= 18{\cdot}9$, third $= 15{\cdot}12$, fourth $= 12{\cdot}6$, fifth $= 10{\cdot}8$. Then the units of work done expansively on one square inch, according to Thomas Simpson's approximate rule, will be

$$= \tfrac{1}{3}\{25{\cdot}2 + 10{\cdot}8 + (18{\cdot}9 + 12{\cdot}6) \times 4 + 15{\cdot}12 \times 2\}$$
$$= 64{\cdot}08.$$

The space through which the piston moves before the steam is cut off $= 3 - \frac{1}{6} = 2\frac{5}{6}$ feet. $\therefore$ the work done before the steam is cut off

$$= 25{\cdot}2 \times 2\tfrac{5}{6} = 71{\cdot}4.$$

Total work of steam on 1 square inch in one stroke

$$= 64{\cdot}08 + 71{\cdot}4 = 135{\cdot}48.$$

$$\text{Mean pressure of the steam} = \frac{135{\cdot}48}{6\frac{5}{6}} = 19{\cdot}82 \text{ lbs.}$$

Since the resistances $=$ the mean pressure of the steam,

$$\therefore \quad \frac{8\ \text{load}}{7} + 1 + 3\tfrac{1}{2} = 19{\cdot}82,$$

$$\therefore \quad \text{load} = 13{\cdot}4 \text{ lbs.}$$

One cubic foot of water generates 1037 cubic feet of steam at 25·2 lbs. pressure.

$\therefore$ The volume of steam discharged a minute

$$= {\cdot}18 \times 1037 = 186{\cdot}66 \text{ cubic feet.}$$

$$\text{Volume discharged each stroke} = \frac{1320}{144} \times 3 = 27{\cdot}5 \text{ cubic feet.}$$

$$\therefore \quad \text{The number of strokes a minute} = \frac{186{\cdot}66}{27{\cdot}5} = 6{\cdot}8.$$

The useful work a minute will be the continued product of the load, the area of the piston, the length of the stroke, and the number of strokes a minute.

∴ the effective horse power

$$=\frac{19{\cdot}82\times1320\times6\tfrac{5}{6}\times6{\cdot}8}{33000'}=24^{h}{\cdot}9.$$

Ques. 18. The area of a piston of a condensing-engine is 4800 square inches; length of stroke, including clearance, 8 feet; the steam cut off at 4 feet; the pressure of the steam 50·4 lbs.; the elasticity of the vapour in the condenser =1·22 lbs.; the clearance=¼ foot; the effective evaporation of the boiler 4·25 cubic feet a minute, and the friction of the unloaded engine 1·2 lbs. to each square inch of piston; required the useful load, and the useful horse power of the engine?

As in the last example, let the space through which the steam acts expansively be divided into four equal parts: first=50·4 lbs., second=40·32, third=33·6, fourth=28·8, fifth=25·2.

Then Simpson's approximate rule gives the units of work done expansively=

$$\tfrac{1}{3}\{50{\cdot}4+25{\cdot}2+(40{\cdot}32+28{\cdot}8)\times4+33{\cdot}6\times2\}$$
$$=139{\cdot}76.$$

In this case the space through which the piston moves before the steam is cut off$=4-\tfrac{1}{4}=3\tfrac{3}{4}$ feet. Hence the units of work done before the steam is cut off

$$=50{\cdot}4\times3\tfrac{3}{4}=189.$$

Total units of work of steam on one square inch in a single stroke$=328{\cdot}76$.

$$\text{Mean pressure}=\frac{328{\cdot}76}{7\frac{3}{4}}=42{\cdot}42\text{ lbs.}$$

$$\therefore\quad \frac{8\times\text{load}}{7}+1{\cdot}22+1{\cdot}2=42{\cdot}42.$$

$$\therefore\quad \text{Load}=35\text{ lbs.}$$

Since a cubic foot of water (see Table) generates 550 cubic feet of steam at 50·4 lbs. pressure, the volume of steam discharged a minute$=4{\cdot}25\times550=2337{\cdot}5$ cubic feet.

$$\text{Volume discharged each stroke}=\frac{4800}{144}\times4=133\tfrac{1}{3}\text{ cubic feet.}$$

∴ The number of strokes a minute$=\frac{2337\cdot5}{133\frac{1}{3}}=17\cdot5$.

Consequently, the effective horse power of this engine must be

$$=\frac{35\times4800\times7\cdot75\times17\cdot5}{33000'}=690\tfrac{5}{11}\text{ HP.}$$

Ques. 19. What must be the evaporation of the boiler of the engine in the last example when the steam has a pressure of 28 lbs., so that the effective horse power may be 300?

By the method employed in the last example the load is found to be 18·5 lbs. Therefore the effective work in one stroke

$$=18\cdot5\times7\cdot75\times4800=688200.$$

The effective units of work in a minute=

$$33000'\times300=9900000'.$$

$$\therefore\ \text{Strokes a minute}=\frac{9900000}{688200}=14\cdot4.$$

Because $133\frac{1}{3}$ cubic feet of steam is discharged in a stroke, the volume of steam discharged a minute$=133\frac{1}{3}\times14\cdot4=1920$ cubic feet, and in this example a cubic foot of water forms 941 cubic feet of steam at 28 lbs. pressure; therefore the cubic feet of water evaporated a minute$=\frac{1920}{941}=2\cdot04$.

Ques. 20. What is the duty of the engine in the last question, supposing a pound of coal to evaporate 7 lbs. of water?

94 lbs., or 1 bushel, evaporates 658 lbs.$=\frac{658}{62\cdot5}$ cubic feet of water.

Useful work a minute$=690\frac{5}{11}\times33000=22785000'$, but this work is done by evaporating 4·25 cubic feet of water.

∴ The work of $\frac{658}{62\cdot5}$ cubic feet, or of one bushel of coal, will be

$$\frac{22785000\times658}{62\cdot5\times4\cdot25}=56444819,$$

or 56 millions units of work the duty of the engine.

The amount of work developed through the expansion of steam may be readily found, with mathematical accuracy, by the following RULE, since the dual logarithm of any given number can be calculated without the use of tables in a few minutes. The art and science of dual arithmetic were invented by Oliver Byrne, the author of the present work; upon this art and science O. B. has written and published five volumes.

RULE.

TO FIND WITH MATHEMATICAL ACCURACY THE ELASTIC FORCE OF STEAM IN UNITS OF WORK.

Multiply the pressure at which the steam is admitted by the distance travelled by the piston before the steam is cut off, which product call (A).

Divide the whole length of the stroke by the above-mentioned distance, and add 100,000000 to the dual logarithm of the quotient; call the sum (B).

Then A multiplied by B gives the whole units of work when eight decimal places are cast off for the dual logarithm.

In question 15, the pressure of the steam=21 lbs., the length of stroke 12 feet, and the steam cut off at 4 feet.

$$4 \times 21 = 84. \quad \text{Call (A).}$$

$$\frac{12}{4} = 3. \quad \text{Dual logarithm of } 3\cdot = \begin{array}{r} 109861229, \\ 100000000, \\ \hline 209861229, \end{array} \text{(B).}$$

$A \times B = 209861229 \times 84 = 176{\cdot}28343236$, the exact area of the figure *f e a d c*, Fig. 33. The approximate rule gave **176·2** the work done on each square inch of piston in a stroke.

In practice the *Indicator* gives a very different figure, something like the dotted line *mp q r n*, Fig. 33. The nearer the Indicator diagram approaches the true or mathematical diagram, the more perfect is the working of the engine. In another part of this work will be found a description of the Indicator.

In the 17th question, the pressure of steam was taken at

48 lbs.; length of stroke=5 ft., and the steam cut off at 2 ft. In this case

$$48 \times 2 = 96, \text{ which call (A).}$$

$$\tfrac{5}{2} = 2{\cdot}5 \text{ dual logarithm of } 2{\cdot}5 = \begin{array}{r} 91629073, \\ \underline{100000000,} \\ 191629073, \text{ (B).} \end{array}$$

$$\mathbf{A \times B = 191629073 \times 96 = 183{\cdot}96391008,}$$

which is mathematically correct. Simpson's approximate rule gave **183·9** on a square inch in a single stroke.

Ques. 21. Steam of 42·35 lbs. pressure is admitted to the cylinder for 2·36 feet of a stroke of 10·45 feet, then being cut off acts expansively; how many units of work is done on one square inch of piston in a single stroke?

$$42{\cdot}35 \times 2{\cdot}36 = 99{\cdot}946 \text{ (A).}$$

$\dfrac{10{\cdot}45}{2{\cdot}36} = 4{\cdot}428$. Dual logarithm of which may be calculated in a few minutes.

$$\text{Dual log. of } 4{\cdot}428 = \begin{array}{r} 148794803, \\ \underline{100000000,} \\ 248794803, \text{ (B).} \end{array}$$

$$\mathbf{A \times B = 99{\cdot}946 \times 248794803 = 248{\cdot}66045380638.}$$

Eight decimals being allowed for the dual logarithm 248794803,

$$99{\cdot}946 \times 2{\cdot}4879 = 2488{\cdot}66,$$

the units of work done on a square inch of piston in a stroke.

To find the proportions in which surcharged and saturated steam combine.

Let a be the weight of the steam used each stroke; temperature, t_1; W, the weight of the surcharged steam; temperature, T; w, the weight of the saturated steam; temperature, t.

$$\therefore \quad W + w = a \quad ; \quad w = a - W,$$

and by a well known property

$$t_1 (W + w) = W\,T + w\,t.$$

$$\therefore \quad W = \frac{a(t_1 - t)}{T - t}$$

Again take the equations

$$W+w=a \quad ; \quad W=a-w, \text{ and}$$

$$t_1 \,(W+w)=W\,T+w\,t$$

$$\therefore \quad t_1\, a=(a-w)\,T+w\,t$$

$$\therefore \quad w=\frac{a\,(T-t_1)}{T-t}$$

$$\therefore \quad W \;:\; w \;::\; t_1-t \;:\; T-t_1$$

EXAMPLE.

A cylinder is half filled with a mixture of saturated and surcharged steam whose temperature on entering=350° Fahr. The cylinder is 96 inches diameter and 10 feet stroke. The temperature of the surcharged steam=580°, and of the saturated 260°; W and w are required?

Area of a circle 96 inches diameter=7238·25

$$\frac{7238{\cdot}25\times 10}{2\times 144}=251{\cdot}33 \text{ cubic feet of}$$

steam in the cylinder half full; 62·5 lbs.=one cubic foot of water. The volume of steam at 350° is 233 times the volume of the water that has produced it;

$$\therefore \quad \frac{251{\cdot}33\times 62{\cdot}5}{233}=67{\cdot}41 \text{ lbs. weight of half the cylinder full}$$

of steam.

T		t_1		t
580	;	350	;	260

$$t_1-t=90^\circ$$

$$T-t_1=230^\circ$$

$$T-t=320^\circ$$

$$W=\frac{67{\cdot}41\times 90}{320}=18{\cdot}95 \text{ lbs. of surcharged steam, and}$$

$$w=\frac{67{\cdot}41\times 230}{320}=48{\cdot}45 \text{ lbs. of saturated steam.}$$

PROOF.

48·45+18·95=67·41 lbs.

18·95 : 48·45 : : 90 : 230.

To find the number of units of heat in steam of any temperature.

Let H be the required units, then, according to Regnault, when the degrees Cent. are reduced to degrees Fahr.,

$$H=1091{\cdot}7+{\cdot}305\ (t^\circ-32^\circ)$$

t° being the given temperature.

EXAMPLES.

Ques. 22. How many units of heat are contained in steam of 432° Fahr.?

$$\begin{array}{r} 432 \\ 32 \end{array}$$

$$1091{\cdot}7+\overline{400}\times{\cdot}305=1213{\cdot}7 \text{ units.}$$

To find the sensible heat of steam when it is equal to the latent heat.

$$1091{\cdot}7+{\cdot}305\ (t^\circ-32^\circ)=2\ t^\circ$$

$${\cdot}305\ t-32\times{\cdot}305=2\ t-1091{\cdot}7$$

$$\therefore\quad t=638^\circ{\cdot}313$$

To find the ratio of the weight of injection water at $60^\circ=t_1$, compared with that of a given weight of steam at temperature $300^\circ=t$; so that the temperature of the mixture may be $=100^\circ=T$.

Let w=a given weight of steam; t its temperature; w_1 the weight of water to be mixed with it; t_1 its temperature; T=the temperature of the mixture.

$$w\ [1091{\cdot}7+{\cdot}305\ (t-32)]+w_1\ t_1=w\ T+w_1\ T$$

$$\therefore\quad \frac{w_1}{w}=\frac{1090{\cdot}4-T+{\cdot}305\ t}{T-t_1}$$

In the example above given,

$$\frac{w_1}{w}=\frac{1090{\cdot}4-100+{\cdot}305+300}{100-60}=27{\cdot}04$$

$$\therefore\quad w_1 : w :: 27{\cdot}04 : 1$$

Ques. 23. If steam of 258° temperature be mixed with steam of 578°·3; the temperature of the mixture to be 334°·4, and the temperature at the exhaust 135°; find the units of heat utilised in the passage of the steam from the boiler to the exhaust both when the steam is mixed and when employed in the ordinary way?

$$T-t=578{\cdot}3-258 \quad =320{\cdot}3$$
$$T-t_1=578{\cdot}3-334{\cdot}4=244{\cdot}9$$
$$t_1-t=333{\cdot}4-258{\cdot} \; = \; 75{\cdot}4$$
$$W : w :: 75{\cdot}4 : 244{\cdot}9$$

$$\begin{array}{r} 244{\cdot}9 \\ 75{\cdot}4 \\ \hline 320{\cdot}3 \text{ lbs.} \end{array}$$

In every 320·3 lbs. of this mixture at 333°·4 Fahr. 75·4 lbs. is surcharged with heat at a temperature of 578°·3, and 244·9 lbs. of steam saturated with water; the saturated steam at the temperature of 258°.

From the Problem, page 85.

$H=1091{\cdot}7+{\cdot}305\,(t^\circ-32)=1183{\cdot}627$ units of heat when $t=333^\circ{\cdot}4$.

$H=1091{\cdot}7+{\cdot}305\,(t^\circ-32)=1123{\cdot}115$ units of heat when $t=135^\circ$.

$$\begin{array}{r} 1183{\cdot}627 \\ 1123{\cdot}115 \\ \hline 60{\cdot}512 \end{array}$$

units of heat utilised in passing from 333°·4 to 135°.

When the same work was done by steam used in the ordinary way, steam of 22·55 lbs. pressure above the atmosphere, or had a pressure$=22{\cdot}55+14{\cdot}7=37{\cdot}25$, which agrees with the temperature 265° nearly; the exhaust steam had a temperature of 133°·6. In steam of temperature 265° there are 1162·765 units of heat, and in steam of temperature 133°·6 there are 1122·688 units of heat.

$$\begin{array}{r} 1162{\cdot}765 \\ 1122{\cdot}688 \\ \hline 40{\cdot}077 \end{array}$$

units of heat utilised in passing from 265° to 133°·6. Consequently, there is an economy of 50 per cent. in favour of the mixture.

Ques. 24. Suppose steam saturated with water of 33·2 lbs. pressure, which answers to 257°·7 temperature, be mixed with steam surcharged with heat, temperature 577°·1; the temperature of the mixture =350°·7, and the temperature of the exhaust =147°·6. In doing the same work, in the same time, with steam in the ordinary way, steam of 34·2 lbs. pressure, or 259°·5 temperature were employed, the temperature of the exhaust being 125°; required the economical advantage in using the mixture?

$$t_1 - t = 350{\cdot}7 - 257{\cdot}7 = 93{\cdot}$$
$$T - t = 577{\cdot}1 - 257{\cdot}7 = 319{\cdot}4$$
$$T - t_1 = 577{\cdot}1 - 350{\cdot}7 = 226{\cdot}4$$
$$\therefore \quad W : w :: 93 : 226{\cdot}4$$

It will be found that 61·9455 units of heat are utilised in passing from 350°·7 to 147°·6. Ordinary steam, in doing the same work in the same time, passes from 259°·5 to 125°, and hence only 41·0225 units of heat are utilised, giving a gain of 40 units in favour of the mixture.

As dual logarithms are readily calculated without the use of tables, the succeeding rules will be found useful when experimental tables are not at hand.

To find the volume of steam of a given pressure that may be produced from a cubic foot of water.

RULE.

From 974096856, the dual logarithm of 17000, take $\frac{40}{43}$ of the dual logarithm of the pressure in lbs.; the corresponding natural number increased by 10, and then added twice to itself consecutively, passing a single figure each time to the right, give the volume of steam.

Ques. 25. How many cubic feet of steam of 55 lbs. pressure on the square inch will a cubic foot of water produce?

Dual logarithm of 55·=400733319,

$$\frac{400733319 \times 40}{43} = 372775181,$$

From 974096856,
Take 372775181,
601321675, = dual log. of 408·79618,

to which add 10. Then,

$$\begin{array}{r} 418{\cdot}79618 \\ 41{\cdot}87962 \\ \hline 460{\cdot}67580 \\ 46{\cdot}06758 \\ \hline \text{Volume of steam} = 506{\cdot}74338 \end{array}$$

∴ A cubic foot of water will make 507 cubic feet of steam of 55 lbs. pressure on the square inch.

Given the temperature of steam to find the corresponding pressure in pounds by dual logarithms.

RULE.

To ·00478822 add 1·8 divided by the temperature plus 4·; call the sum (A).

From the quotient of 8827643, divided by (A), take the whole number 403938299, the remainder is the dual logarithm of the pressure in pounds on the square inch.

Ques. 26. When steam has a temperature of 400° Fahr., what is the corresponding pressure in pounds on the square inch?

$$\frac{1{\cdot}8}{404{\cdot}} = {\cdot}00445544$$

$$\begin{array}{rl} {\cdot}00478822 & \text{constant} \\ \hline {\cdot}00924366 & \end{array}$$

$$\frac{8827643,}{{\cdot}00924366} = 954994342, \text{ a whole number}$$

From 954994342,
Take 403938299, constant
551056043, = the dual logarithm of 247·289668

∴ 247·3 lbs. corresponds to the temperature of 400° Fahr.

TABLE OF THE ELASTIC FORCE OF STEAM, AND THE CORRESPONDING TEMPERATURE; THE STEAM BEING IN CONTACT WITH THE WATER THAT PRODUCED IT.

Pressure on a square inch, including the pressure of the Atmosphere.		Elastic force in		Temperature in degrees of			Volume of Steam, Volume of Water =1.
		Inches of Mercury.	Metres of Mercury.	Fahr.	Reau.	Cent.	
lbs.	kilog.						
14·7	6·668	30·00	·762	212·0	80·0	100·0	1700
15	6·80	30·60	·778	212·8	80·4	100·4	1669
16	7·26	32·64	·829	216·3	81·9	102·4	1573
17	7·71	34·68	·880	219·6	83·3	104·2	1488
18	8·16	36·72	·932	222·7	84·7	105·9	1411
19	8·62	38·76	·984	225·6	86·0	107·6	1343
20	9·07	40·80	1·037	228·5	87·3	109·2	1281
21	9·52	42·84	1·089	231·2	88·5	110·7	1225
22	9·98	44·88	1·140	233·8	89·7	112·1	1174
23	10·43	46·92	1·192	236·3	90·8	113·5	1127
24	10·88	68·96	1·244	238·7	91·9	114·8	1084
25	11·34	51·00	1·296	241·0	93·0	116·1	1044
26	11·79	53·04	1·348	243·3	93·9	117·4	1007
27	42·25	55·08	1·400	245·5	94·9	118·6	973
28	12·70	57·12	1·452	247·6	95·8	119·8	941
29	13·15	59·16	1·503	249·6	96·7	120·9	911
30	13·61	61·21	1·555	251·6	97·6	122·0	883
31	14·06	63·24	1·607	253·6	98·5	123·1	857
32	14·51	65·28	1·659	255·5	99·3	124·2	833
33	14·97	67·32	1·711	257·3	100·1	125·2	810
34	15·42	69·36	1·763	259·1	100·9	126·2	788
35	15·87	71·40	1·814	260·9	101·7	127·2	767
36	16·33	73·44	1·866	262·6	102·5	128·1	748
37	16·78	75·48	1·918	264·3	103·2	129·1	729
38	17·23	77·52	1·970	265·9	104·0	129·9	712
39	17·69	79·56	2·022	267·5	104·7	130·8	695
40	18·14	81·60	2·074	269·1	105·4	131·7	679
41	18·59	83·64	2·126	270·6	106·0	132·6	664
42	19·05	85·68	2·178	272·1	106·7	133·4	649
43	19·50	87·72	2·229	273·6	107·4	134·2	635
44	19·96	89·76	2·281	275·0	108·0	135·0	622
45	20·41	91·80	2·333	276·4	108·6	135·8	610
46	20·86	93·84	2·385	277·8	109·2	136·6	598
47	21·32	95·88	2·437	279·2	109·9	137·3	586
48	21·77	97·92	2·489	280·5	110·4	138·1	575

TABLE—*continued.*

Pressure on a square inch, including the pressure of the Atmosphere.		Elastic force in		Temperature in degrees of			Volume of Steam, Volume of Water =1.
lbs.	kilog.	Inches of Mercury.	Metres of Mercury.	Fahr.	Reau.	Cent.	
49	22·22	99·96	2·541	281·9	111·1	138·8	564
50	22·68	102·00	2·592	283·2	111·6	139·6	554
51	23·13	104·04	2·644	284·4	112·2	140·2	544
52	23·59	106·08	2·696	285·7	112·8	140·9	534
53	24·04	108·12	2·748	286·9	113·3	141·6	525
54	24·49	110·16	2·800	288·1	113·8	142·3	516
55	24·95	112·20	2·852	289·3	114·4	142·9	508
56	25·40	114·24	2·903	290·5	114·9	143·6	500
57	25·85	116·28	2·955	291·7	115·4	144·3	492
58	26·31	118·32	3·007	292·9	116·0	144·9	484
59	26·76	120·36	3·059	294·2	116·5	145·7	477
60	27·21	122·40	3·111	295·6	117·2	146·4	470
61	27·67	124·44	3·163	296·9	117·7	147·2	463
62	28·12	126·48	3·215	298·1	118·3	147·8	456
63	28·57	128·52	3·266	299·2	118·8	148·4	449
64	29·03	130·56	3·318	300·3	119·2	149·1	443
65	29·48	132·60	3·370	301·3	119·7	149·6	437
66	29·93	134·64	3·422	302·4	120·1	150·2	431
67	30·39	136·68	3·474	303·4	120·5	150·8	425
68	30·84	138·72	3·526	304·4	121·9	151·3	419
69	31·29	140·76	3·577	305·4	121·3	151·9	414
70	31·75	142·80	3·629	306·4	122·7	152·4	408
71	32·20	144·84	3·681	307·4	122·1	153·0	403
72	32·66	146·88	3·733	308·4	122·5	153·6	398
73	33·11	148·92	3·785	309·3	123·9	154·1	393
74	33·56	150·96	3·837	310·3	123·3	154·6	388
75	34·02	153·02	3·889	311·2	124·7	155·1	383
76	34·47	155·06	3·940	312·2	124·0	155·7	379
77	34·93	157·10	3·992	313·1	124·4	156·2	374
78	35·38	159·14	4·044	314·0	125·3	156·7	370
79	35·83	161·18	4·096	314·9	125·7	157·2	366
80	36·29	163·22	4·148	315·8	126·1	157·7	362
81	36·74	165·26	4·199	316·7	126·5	158·2	358
82	37·19	167·30	4·252	317·6	126·9	158·7	354
83	37·65	169·34	4·303	318·4	127·3	159·1	350
84	38·10	171·38	4·355	319·3	127·7	159·6	346
85	38·55	173·42	4·407	320·1	128·0	160·1	342

TABLE—*continued.*

Pressure on a square inch, including the pressure of the Atmosphere.		Elastic force in		Temperature in degrees of			Volume of Steam, Volume of Water =1.
		Inches of Mercury.	Metres of Mercury.	Fahr.	Reau.	Cent.	
lbs.	kilog.						
86	39·01	175·46	4·459	321·0	128·4	160·6	339
87	39·46	177·50	4·511	321·8	128·8	161·0	335
88	39·91	179·54	4·563	322·6	129·2	161·4	332
89	40·37	181·58	4·615	323·5	129·6	161·9	328
90	40·82	183·62	4·666	324·3	129·9	162·4	325
91	41·27	185·66	4·718	325·1	130·3	162·8	322
92	41·73	187·70	4·770	325·9	130·6	163·3	319
93	42·18	189·74	4·822	326·7	131·0	163·7	316
94	42·64	191·78	4·874	327·5	131·3	164·2	313
95	43·09	193·82	4·926	328·2	131·6	164·8	310
96	43·54	195·86	4·977	329·0	132·0	165·0	307
97	44·00	197·90	5·029	329·8	132·4	165·4	304
98	44·45	199·92	5·081	330·5	132·7	165·8	301
99	44·90	201·96	5·133	331·3	133·0	166·3	298
100	45·36	204·01	5·185	332·0	133·3	166·7	295
110	49·89	224·40	5·703	339·2	136·5	170·7	271
120	54·43	244·82	6·222	345·8	139·5	174·3	251
130	58·97	265·23	6·740	352·1	142·3	177·8	233
140	63·50	285·61	7·259	357·9	144·8	181·1	218
150	68·04	306·03	7·778	363·4	147·3	184·1	205
160	72·57	326·42	8·296	368·7	149·6	187·1	193
170	77·11	346·80	8·814	373·6	151·8	189·8	183
180	81·65	367·25	9·333	378·4	153·9	192·4	174
190	86·18	387·61	9·851	382·9	156·0	194·9	166
200	90·72	408·04	10·370	387·3	157·9	197·4	158

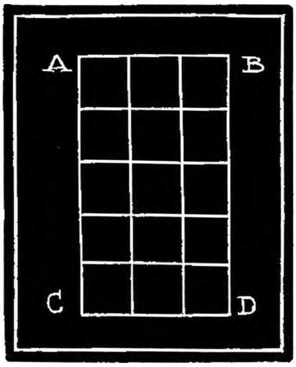
CHAPTER VI.

Accumulated Work.

If a body moves uniformly, the space described is equal to the product of the time by the velocity.

A body moving 3 ft. a second will pass over 15 ft. in 5 seconds.

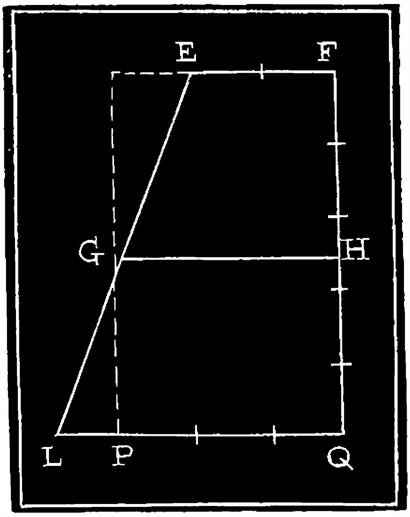
Fig. 35.



Therefore, if a rectangle like A B C D be constructed, C D=3 the units in the velocity, the perpendicular B D = 5 the units of time; then the small squares in the rectangle A D answer as counters to designate 15 feet in length.

In a similar way, the space described by a body whose motion is uniformly accelerated may be told by the area of a trapezoid, whose parallel sides contain respectively the units of velocity at the commencement and end of the motion, the perpendicular between those sides being the units in the intervening time. This last proposition will be easily understood, since the mean breadth of the trapezoid contains the same number of units as the velocity which the body has at the middle of the time. This method of making the area of figure represent a distance is useful, and should be carefully examined by a young student.

Fig. 36.



Let E F = velocity of 2 feet a second, then suppose the body uniformly accelerated for 5 seconds; at the end suppose its velocity to be 4 feet a second. If L Q = 4 feet, F Q = 5 units from the same or any other scale of equal parts. The area of E F Q L in small squares or parallelograms may be used as counters to designate the space passed over

$$\frac{2+4}{2}=3=\text{G H}.$$

$5 \times 3 = 15$ ft. space passed over.

In this way the length of a straight line is represented by a plane area. The best illustration of accelerated motion is afforded in the case of falling bodies near the surface of the earth.

FORCE OF GRAVITY NEAR THE SURFACE OF THE EARTH.

Bodies being allowed to fall freely near the surface of the earth, the force of the attraction being constant, will communicate to them equal additions of velocity in equal intervals of time. Thus, at the end of one second, the velocity of the body is 32·155 feet; at the end of two seconds, 2 times 32·155 feet; at the end of 3 seconds, 3 times 32·155 feet; at the end of 4 seconds, 4 times 32·155 feet, and so on; or generally, the velocity acquired by a falling body is equal to the product of the time of the body's fall in seconds by $32\frac{1}{6}$ feet, putting $32\frac{1}{6}$ for 32·155.

This is expressed by the equation,

$$v = t \times 32\tfrac{1}{6}.$$

The space described by a body in one second will be half of $32\frac{1}{6}$ feet $= 16\frac{1}{12}$ feet; because the velocity of the body in the middle of the time will be the mean velocity with which it moves during that time. In like manner, the space described by the body in 4 seconds, will be 4 times $2 \times 32\frac{1}{6}$ feet; because, $4 \times 32\frac{1}{6}$ feet is the velocity at the end of the time, and, therefore, $2 \times 32\frac{1}{6}$ will be the mean velocity, or the velocity in the middle of the time. But 4 times $2 \times 32\frac{1}{6} = 16 \times 16\frac{1}{12} = 4^2 \times 16\frac{1}{12}$, that is, the space described by a falling body in 4 seconds is equal to the square of the time, multiplied by the space described in one second. In the same manner, any other case may be established. In general, the relation of the space S in feet, and the time t in seconds, is expressed by the equation

$$S = t^2 \times 16\tfrac{1}{12}.$$

Ques. 1. What velocity will a falling body have at the end of 5 seconds?

$$32\tfrac{1}{6} \times 5 = 160\tfrac{5}{6} \text{ feet.}$$

Ques. 2. In what time will a body acquire a velocity of 193 feet a second?

$$\frac{193}{32\frac{1}{6}} = 6 \text{ seconds.}$$

Ques. 3. Through what space will a body fall in $4\frac{1}{2}$ seconds?

$$(4\tfrac{1}{2})^2 \times 16\tfrac{1}{12} = 325\tfrac{11}{16} \text{ feet.}$$

Ques. 4. A body is thrown downward with a velocity of 10 feet per second, how far will it descend in 6 seconds?

It is evident that the body will retain its motion of projection, although acted upon by gravity. Now the space due to the projection $= 6 \times 10 = 60$ feet, and this added to the space due to gravity will give

$$60 + 6^2 \times 16\tfrac{1}{12} = 639 \text{ feet.}$$

Ques. 5. In what time will a body fall 193 feet?

By the general equation

$$t^2=\frac{193}{16\frac{1}{12}}=12.$$

$$\therefore \quad t=\sqrt{12}=3{\cdot}464 \text{ seconds.}$$

Ques. 6. From what height must a body fall to acquire a velocity of $96\frac{1}{2}$ feet?

First it is necessary to find the time which the body will have to fall, to acquire a velocity of $96\frac{1}{2}$ feet.

$$\frac{96\frac{1}{2}}{32\frac{1}{6}}=3 \text{ seconds.}$$

$$\text{Height}=3^2\times16\frac{1}{12}=144\frac{3}{4} \text{ feet.}$$

Ques. 7. The velocity of a body is 84 feet. From what height would it have to fall, in order to attain this motion?

$$\text{Time}=\frac{84}{32\frac{1}{6}}; \text{ distance or space}$$

$$=\left(\frac{84}{32\frac{1}{6}}\right)^2\times16\frac{1}{12}=\frac{84^2}{2\times32\frac{1}{6}}=109{\cdot}7 \text{ feet.}$$

Put $g=32\frac{1}{6}$ and $a=84$, then

$$\frac{a^2}{g^2}\times\frac{g}{2}=\frac{a^2}{2g}$$

∴ The space fallen through to acquire the velocity a is equal to the square of (a,) divided by ($2\,g$). In most modern works, g is put for the number $32\frac{1}{6}$, and π is put for the circumference of a circle when the diameter = 1, or π = arc of a semicircle, radius = 1.

OF THE WORK ACCUMULATED BODY.

When a body is in motion, it will continue in that state unless acted upon by some external force. But in order to give this motion to the body, there must be work done upon it. A velocity of $32\frac{1}{6}$ feet may be given to one pound, by raising it $16\frac{1}{12}$ feet, and then allowing it to fall by the force of gravity.

In this case, the units of work accumulated in the body will be

= ☞ $16\frac{1}{12}$.

As another familiar example, take a fly-wheel in rapid motion; a portion of the work of the engine must have gone to produce this motion, and before the engine can come to a state of rest, all the work accumulated in the fly, as well as in the other parts of the machine, must be destroyed. In this way a fly-wheel acts as a reservoir of work.

IN ORDER TO ESTIMATE THE WORK IN A MOVING BODY, IT IS SIMPLY NECESSARY TO CONSIDER THE HEIGHT FROM WHICH IT MUST FALL TO ACQUIRE THE GIVEN VELOCITY, AND THEN THE WORK WILL BE FOUND BY MULTIPLYING THAT HEIGHT IN FEET BY THE WEIGHT OF THE BODY IN POUNDS; BECAUSE THE WORK EXPENDED IN RAISING THE BODY TO THE NECESSARY HEIGHT TO COMMUNICATE THE GIVEN VELOCITY MUST BE THE SAME AS THE WORK, WHICH GRAVITY WILL PERFORM UPON THE BODY IN ITS DESCENT.

Ques. 8. The weight of a ram is 600 lbs., and at the end of the blow has a velocity of $32\frac{1}{6}$ feet; what work has been done in raising it?

The ram must have fallen $16\frac{1}{12}$ feet; the work done upon it therefore

$$=600\times16\tfrac{1}{12}=9650.$$

Ques. 9. How many units of work are accumulated in a body whose weight is 144 lbs., and velocity 200 feet?

The height from which the body must fall, to acquire the given velocity,

$$=\frac{200^2}{2\times32\frac{1}{6}}=621{\cdot}76 \text{ feet.}$$

Consequently, the work which must have been done upon the body

$$=621{\cdot}76\times144=89533{\cdot}68.$$

Ques. 10. Required the work accumulated in a cannon-ball whose weight is $32\frac{1}{6}$ lbs., and velocity of 1500 feet?

The height from which the ball must fall, to acquire a velocity of 1500,

$$=\frac{1500^2}{2\times 32\frac{1}{6}};$$

The work accumulated in the body

$$=\frac{1500^2\times 32\frac{1}{6}}{2\times 32\frac{1}{6}}=1125000.$$

This example clearly shows, that *the work accumulated in a moving body is equal to the square of the velocity in feet per second, multiplied by the weight of the body in lbs., and divided by* $2\times 32\frac{1}{6}$.

This very important proposition is expressed by the equation

$$U=\frac{V^2\times W}{2\,g}.$$

Ques. 11. A ball weighing 20 lbs. is projected with a velocity of 60 feet per second, on a bowling-green. What space will the ball move over before it comes to rest, allowing the friction to be $\frac{1}{18}$ the weight of the ball?

The units of work, U, in the ball

$$=U=\frac{60^2\times 20}{2\times 32\frac{1}{6}}=1119{\cdot}17.$$

It is evident that the ball will not stop until all this work is destroyed—that is, until the work destroyed by friction is equal to the accumulated work.

Let x be the number of feet over which the ball moves before it comes to a state of rest, then the work destroyed by friction in moving the ball over x feet

$$=\frac{20}{18}\times x=1119{\cdot}17;$$

$$\therefore\quad x=\frac{1119{\cdot}17\times 18}{20}=1007{\cdot}3 \text{ feet.}$$

Ques. 12. A train weighs 193 tons, and has a velocity of 30 miles an hour when the steam is turned off; how far will the train move upon a level rail, whose friction is $15\frac{1}{2}$ lbs. per ton?

30 miles an hour = 44 feet a second, for

$$\frac{30 \times 5280}{60 \times 60} = 44.$$

$$193 \text{ tons} = 432320 \text{ lbs.}$$

Putting **U** for the units of work,

$$\therefore \quad \mathbf{U} = \frac{44^2 \times 432320}{2 \times 32\frac{1}{6}} = 13009920.$$

When the train stops, the work of friction will be equal to the work accumulated in the train, hence, if x be the number of feet the train will move after the steam is cut off, the work in moving the train over x feet

$$= 193 \times 5\tfrac{1}{2} \times x = \mathbf{U}.$$

$$\therefore \; 193 \times 5\tfrac{1}{2} \times x = 13009920.$$

$$\therefore \quad x = \frac{13009920}{193 \times 5\frac{1}{2}} = 12256 \text{ feet.}$$

Ques. 13. A train weighing 60 tons has a velocity of 40 miles an hour when the steam is turned off, how far will it ascend an incline of 1 in 100, taking friction at 8 lbs. a ton.

40 miles an hour = $58\frac{2}{3}$ feet a second.

$$60 \text{ tons} = 134400 \text{ lbs.}$$

Work in the train when the steam is cut off

$$= \frac{(58\frac{2}{3})^2 \times 134400}{2 \times 32\frac{1}{6}};$$

put x = the number of feet the train will move after the steam is turned off; then the work due to the friction

$$= 60 \times 8 \times x = 480\, x;$$

$$100 : 1 :: x : \frac{x}{100} =$$

the rise of the rail in x feet;

$\therefore$ The work due to gravity

$$=\frac{x}{100}\times 134400=1344\,x;$$

$$\therefore\quad 480\,x+1344\,x=\frac{(58\frac{2}{3})^2\times 134400}{2\times 32\frac{1}{6}};$$

$$\therefore\quad x=\frac{(58\frac{2}{3})^2\times 134400}{1824\times 64\frac{1}{3}}=3942\tfrac{1}{2}\text{ feet.}$$

Ques. 14. Two balls, each weighing $64\frac{1}{3}$ lbs., are placed at the extremities of a horizontal arm, which gives motion to a screw driving a punch, as in the common stamping machine. The velocity given to the balls is 8 feet a second. What is the mean resistance opposed to the punch, when it is just driven through an iron plate $\frac{3}{8}$ of an inch thick?

If the area of the cross section of the punch $=\frac{1}{1000}$ of a square inch, the pressure on the square inch $=4096$ lbs.

The weight of the two balls $=128\frac{2}{3}$ lbs.

If the velocity of the balls be 8 feet, just as the punch begins to cut, then the units of work accumulated

$$=U=\frac{8^2\times 128\frac{2}{3}}{2\times 32\frac{1}{6}}=128.$$

Now, if the thickness of the plate be 1 foot, the uniform resistance would obviously be 128 lbs.; but the thickness is $\frac{1}{32}$ of a foot.

$$\therefore\quad \text{Mean resistance, }\frac{128}{\frac{1}{32}}=4096\text{ lbs.}$$

Ques. 15. A ball weighs 24 lbs., is fired from the mouth of a cannon 12 feet long, with the velocity of 1000 feet a second; it is required to find the mean pressure of the elastic vapour upon the ball?

$$U=\frac{1000^2\times 24}{2\times 32\frac{1}{6}}=373057.$$

Let x be the pressure upon the ball during the passage through the barrel, then

$$12 \times x = 373057.$$

$$\therefore \quad x = 31088 \text{ lbs. mean pressure.}$$

The area of a circular cross section of a 24 lb. cast-iron round shot=24 square inches;

$$\therefore \quad \frac{31088}{24} = 1295\tfrac{1}{3} \text{ lbs.,}$$

the pressure on each square inch of cross section of bore.

Ques. 16. A carriage of 1 ton moves on a level rail with the speed of 8 feet a second; through what space must the carriage move to have a velocity of 2 feet, supposing friction to be 6 lbs. a ton?

$$\text{Work lost} = \frac{8^2 \times 2240}{64\frac{1}{3}} - \frac{2^2 \times 2240}{64\frac{1}{3}} = 2089.$$

But this work has been taken up by friction passing over a space which put$=x$ feet, then

$$x \times 6 = 2089;$$

$$\therefore \quad x = 348 \text{ feet.}$$

Ques. 17. If the carriage in the last example move over 400 feet before it comes to a state of rest, what is the resistance of friction per ton?

Work accumulated in the carriage

$$= \frac{8^2 \times 2240}{64\frac{1}{3}} = 2228;$$

$$\text{Work of friction} = \text{friction} \times 400;$$

$$\therefore \quad \text{Friction} \times 400 = 2228;$$

$$\text{Friction} = 5{\cdot}57 \text{ lbs.}$$

Ques. 18. Two weights, W, w, weighing 7 and 4 lbs. respectively, are connected by a cord that goes over a fixed pulley, as in Atwood's machine; through what space must W descend to acquire a velocity of 3 ft. a second?

As the velocity with which w ascends is the same as that with which W descends, then

$$\text{Work in } \mathbf{W} = \frac{3^2 \times 7}{64\frac{1}{3}} = \cdot 9793;$$

$$\text{Work in } w = \frac{3^2 \times 4}{64\frac{1}{3}} = \cdot 1865.$$

$$\text{Total accumulated work} = 1 \cdot 1658.$$

Let x be the space passed over by each of the weights, then

$$\text{Work of gravity on W} = 7 \times x$$

$$\text{Work of gravity on } w = 4 \times x$$

$$3 \times x \text{ difference.}$$

As the work performed on w has been produced by the work of W; ∴ the work existing in the bodies must be equal, or

$$3 \times x = 1 \cdot 1658.$$

$$\therefore \quad x = \cdot 3886 \text{ feet} = 4\tfrac{2}{3} \text{ inches, nearly.}$$

Ques. 19. A weight of 6 lbs. acts upon a weight of 4 lbs. over a pulley; in what time will it descend $115\frac{4}{5}$ feet, and what will be the common velocity of the bodies at the end of that time?

$$6 - 4 = 2 \text{ lbs.}$$

Units of work in both bodies $= 115\frac{4}{5} \times 2 = 231\frac{3}{5}$.

Then, if V be the velocity, the units of work in both bodies will also be

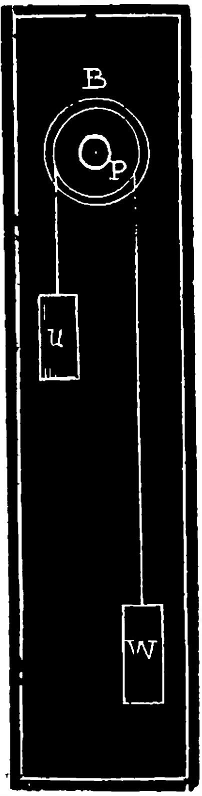
$$= \frac{V^2}{64\frac{1}{3}} \times 10 = 231\tfrac{3}{5}.$$

$$\therefore \quad V = 38\tfrac{3}{5} \text{ feet velocity at the end of } 115\tfrac{4}{5} \text{ feet.}$$

$$\text{Mean velocity} = 19\tfrac{3}{10}.$$

$$\frac{115\frac{4}{5}}{19\frac{3}{10}} = 6 \text{ seconds the time of descent.}$$

Fig. 37.



Ques. 20. Two weights, W, *w*, Fig 37, of 18 and 16 ounces respectively, are suspended by a fine silk thread going over a cast-iron pulley, B; how long will W be descending 20 feet, and what velocity will it have acquired, allowing for the friction of the brass pin, P; the circumference of the hole, P=1 inch, and of the pulley, B=1 foot?

When W has moved 20 feet, the surfaces of P have been in contact over a space of 20 inches$=1\frac{2}{3}$ feet. According to Morin, the co-efficient of friction upon brass axles, in cast-iron bearings, grease being continuously renewed$=\cdot 045$;

$$18+16 \text{ ounces}=2\tfrac{1}{8}\text{ lbs.}$$

Units of work destroyed by the friction of the pin=

$$1\tfrac{2}{3}\times 2\tfrac{1}{8}\times\cdot 045=\cdot 159375.$$

$$18-16 \text{ ounces}=\tfrac{1}{8}\text{ lb.}$$

$\frac{1}{8}\times 20=2\cdot 5$ developed.
$\cdot 159375$ due to friction.
$2\cdot 340625$

Let V be the required velocity, then

$$\frac{V^2}{64\frac{1}{3}}\times 2\tfrac{1}{8}=2\cdot 340625.$$

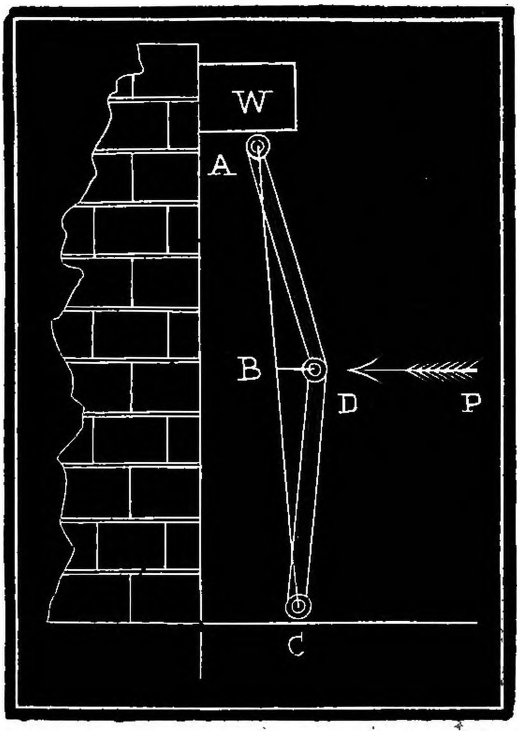
$$\therefore\quad V^2=69\cdot 915.$$

$$V=8\cdot 36 \text{ feet velocity.}$$

$$\frac{20\cdot}{4\cdot 18}=4\cdot 78 \text{ seconds time occupied.}$$

Fig. 38.

Ques. 21. Let A D=D C=20 inches, arms of a toggle joint; P, a pressure of 200 lbs. applied at D in the direction of the perpendicular B D=1 inch; when the two bars W D, D C, are brought into a straight line, what weight, W, may be moved, supposing the point, C, stationary?



$$\sqrt{20^2-1}=\sqrt{399}=19{\cdot}9749844=\text{A B}.$$

$$\therefore \quad \text{A C}=2\text{ A B}=39{\cdot}9499688.$$

W will be raised the difference between A D+D C, and A C= 0·0500312 inches, or $\frac{\cdot 0500312}{12}$ feet, while P moves over D B=1 inch=$\frac{1}{12}$ feet.

Units of work done by P=$\frac{1}{12}\times 200$.

Units of work done in raising W=

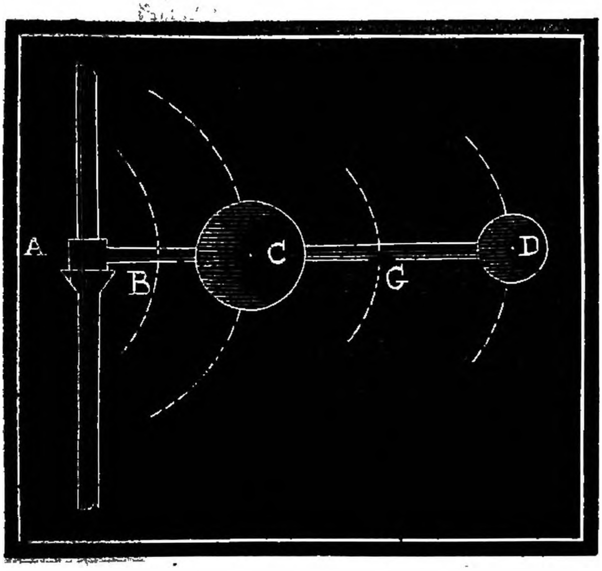
$$\frac{\cdot 0500312}{12}\times \text{W}=\frac{1}{12}\times 200.$$

$$\text{W}=3997{\cdot}5 \text{ lbs.}$$

The friction of the joints are not taken into account.

HOW TO ESTIMATE THE UNITS OF WORK IN A ROTATING BODY.

Fig. 39.



To take a simple case, let two balls, C and D, be connected by a rod, A D, and made to revolve round the centre, A; suppose C to weigh 50 lbs. and D 20 lbs.; the distance of C from A = 12 feet, and that of D = 27 feet. It is required to find the units of work in these balls when the point B, 1 foot from the centre of motion A, moves at the rate of 19·3 feet a second.

The *centre of gyration* is also required—that is a point, G, in the rod where we may suppose the weight of the two balls collected—so that the amount of work may remain the same as when the bodies were apart.

Velocity of C = (19·3 × 12) feet a second.
Velocity of D = (19·3 × 27) feet a second.

$$\text{Units of work in C} = \frac{(19{\cdot}3 \times 12)^2 \times 50}{64\frac{1}{3}} = 41688.$$

$$\text{Units of work in D} = \frac{(19{\cdot}3 \times 27)^2 \times 20}{64\frac{1}{3}} = 84418{\cdot}2.$$

$$\text{Total units of work} = 126106{\cdot}2.$$

Let x be the distance, A G, then the work in the two balls collected at G =

$$\frac{(19{\cdot}3x)^2 \times 70}{64\frac{1}{3}} = 126106{\cdot}2.$$

$$\therefore \quad x^2 = 311\tfrac{1}{7}, \text{ and } x = 17{\cdot}639 \text{ feet.}$$

Now, if the two weights 20+50=70 lbs. be placed on the rod at the *centre of gyration*, G, and move with a uniform velocity of (17·639 × 19·3) feet a second, the amount of work in the bodies thus combined is the same as when posited at C and D.

From this simple case it is evident that when the *centre of gyration* of a rotating body is known, the accumulated work in that body is readily found. To find the *centre of gyration* in differently formed bodies requires the aid of a higher calculus, the introduction of which would be out of place in the present work. However, it is necessary to observe that the distance of this centre from the axis of rotation in a circular wheel of uniform thickness is equal to the radius of the wheel × $\sqrt{\frac{1}{2}}$; in a rod revolving about its extremity it is equal to the length of the rod × $\sqrt{\frac{1}{3}}$, and when the rod revolves about its centre it is equal to the length × $\sqrt{\frac{1}{12}}$; and in a plane rim, like the rim of a fly-wheel, it is equal to the square root of one-half the sum of the squares of the radii forming the ring.

Ques. 22. The weight of a fly-wheel=8000 lbs., suppose the centre of gyration to be 10 ft. from the axis, the diameter of which =14 inches; the wheel makes 27 revolutions a minute, how many revolutions will it make before it stops, the friction of the axis being $\frac{1}{5}$ of the whole weight?

Velocity of centre of gyration a second=

$$\frac{20 \times 3{\cdot}1416 \times 27}{60} = 28{\cdot}2744 \text{ ft.}$$

$$\text{Work in the wheel} = \frac{(28{\cdot}2744)^2 \times 8000}{64\frac{1}{3}} = 99411{\cdot}77.$$

Circumference of the axis in feet =

$$\frac{14}{12} \times 3{\cdot}1416 = 3{\cdot}6652.$$

Work destroyed in x revolutions=

$$3{\cdot}6652 \times x \times \frac{8000}{5} = 5864{\cdot}32\ x.$$

$$\therefore \quad x=\frac{99411\cdot 77}{5864\cdot 32}=16\cdot 952 \text{ revolutions.}$$

Ques. 23. The weight of a fly-wheel is $1\frac{1}{2}$ tons, the distance of the *centre of gyration* from the axis = 8 feet, and the number of revolutions a minute=24; what number of strokes will this wheel give 2 forge hammers each weighing 250 lbs., each hammer having a lift of 3 ft., friction being neglected?

Velocity of the centre of gyration=20·1 ft. a second. Work in the wheel=

$$\frac{(20\cdot 1)^2\times 3360}{64\frac{1}{3}}=22984\cdot 6.$$

Work of x lifts of the hammers=

$$250\times 2\times 3\times x=1500\,x$$

$$\therefore \quad 1500\,x=22984\cdot 6$$

$$\therefore \quad x=15\cdot 3 \text{ lifts}$$

to each hammer.

Ques. 24. The diameter of a grindstone is 5·6 ft., and its weight=386 lbs., the circumference is made to revolve with a velocity of 6 ft. a second; the circumference of the axis=8 inches, the friction of it=$\frac{1}{7}$ of the weight; find the number of revolutions made by the stone when left to itself?

The *centre of gyration* from the axis of the grindstone $=2\cdot 8\sqrt{\frac{1}{2}}$.

To find the velocity of the centre of gyration,

$$\frac{5\cdot 6}{2} : 6 :: 2\cdot 8\sqrt{\tfrac{1}{2}} : 6\sqrt{\tfrac{1}{2}};$$

The square of $6\sqrt{\frac{1}{2}}=18$.

$$\text{Work in the stone}=\frac{18\times 386}{64\frac{1}{3}}=108.$$

Let x be the number of revolutions,

then $\frac{8}{12}\times\frac{386}{7}\times x=$ the work destroyed in x revolutions.

$$\therefore \quad \frac{8}{12} \times \frac{386}{7} \times x = 108.$$

$\therefore \quad x = 6$ revolutions, nearly.

Ques. 25. How many units of work are developed by the sledge-hammer referred to in question 28, page 50?

Units of work in one blow=

$\dfrac{(38{\cdot}6)^2 \times 30}{64\frac{1}{3}} = 694{\cdot}8$, hence 20 blows will develop **13896** units of work.

Ques. 26. Supposing the earth to be a perfect sphere four times the weight of the size of itself of water, radius $= 20900000$ feet $= r$; how many steam engines of 1600^{h} each, applied in the direction of the centre of gyration, would be required to stop the earth revolving on its axis in an hour and forty minutes.

$$1600^{h} = 52800000'.$$

Units of work in 100 minutes $= 5280000000$.

$\dfrac{\pi}{6}(2\, r)^3 \times 62{\cdot}5 \times 4 = \dfrac{1000\, \pi\, r^3}{3} =$ the weight of the earth in lbs. This weight divided by $29 = 64\frac{1}{3}$ gives $\dfrac{1000\, \pi\, r^3}{193}$.

$$\pi = 3{\cdot}1459265$$

The distance of the *centre of gyration* from the centre of the sphere $= r \sqrt{}$ Fig. 43. $2\, r \sqrt{\frac{2}{5}} \times \pi =$ the circumference of the circle described by the centre of gyration in 24 hours.

$\therefore \quad \dfrac{2\, \pi\, r \sqrt{\frac{2}{5}}}{24 \times 60 \times 60} =$ the velocity of the centre of gyration in feet a second. Velocity squared $= \dfrac{4\, \pi^2\, r^2 \times \frac{2}{5}}{7464960000}$. Then the units of work in the globe will be

$$\frac{4\, \pi^2\, r^2 \times \frac{2}{5}}{7464960000} \times \frac{1000\, \pi\, r^3}{193} = \frac{8\, \pi^3\, r^5}{7464960 \times 965}$$

These units of work divided by **5280000000** gives

$$\frac{\pi^3\ r^5}{7464960\times 965\times 660000000}=2065772000000000000000,$$

nearly, the number of steam engines required.

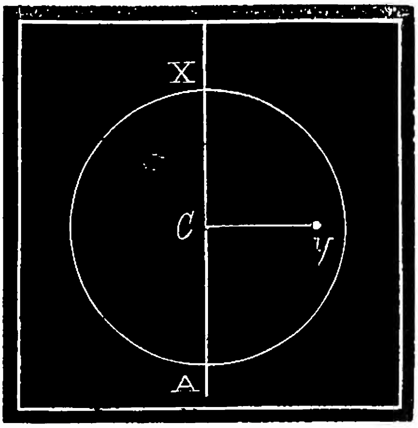
Let y be the centre of gyration in the following bodies, A X the axis about which each body revolves, C y the perpendicular distance from the centre to the axis of rotation.

In a mere wire circumference, Fig. 40, revolving about its diameter, A X,

$$C\,y=C\,X\ \sqrt{\tfrac{1}{2}}=\cdot 7071\ r,$$

r being put for C X or C A, the radius.

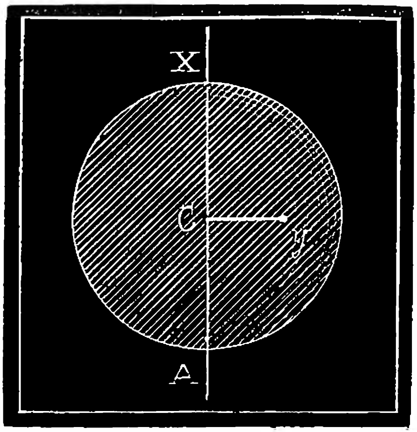
Fig. 40.



In a circular plate, Fig. 41, revolving round its diameter,

$$C\,y=\tfrac{1}{2}\text{ of }C\,X=\cdot 5\ r.$$

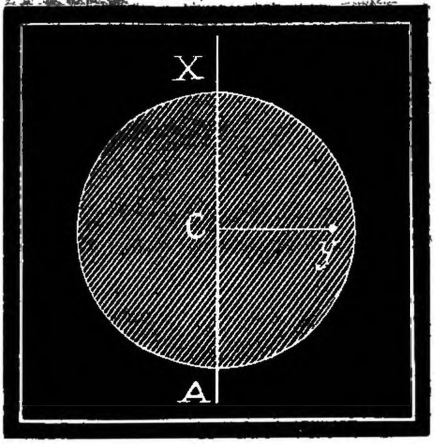
Fig. 41.



In a very thin hollow globe, Fig. 42,

$$C\,y = C\,X \text{ multiplied by } \sqrt{\tfrac{2}{3}} = \cdot 8165\,r,$$
$$r = C\,X = C\,A.$$

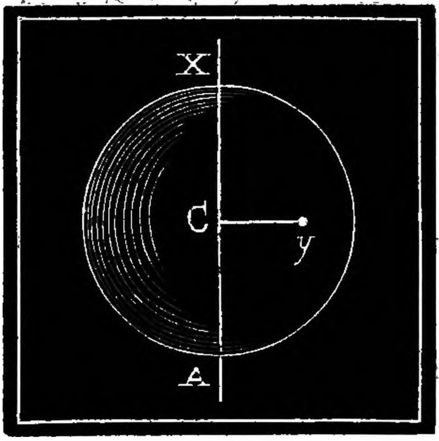
Fig. 42.



In a solid sphere, Fig. 43,

$$C\,y = C\,X \text{ multiplied by the square root of } \tfrac{2}{5} = \cdot 63245553 \times r.$$

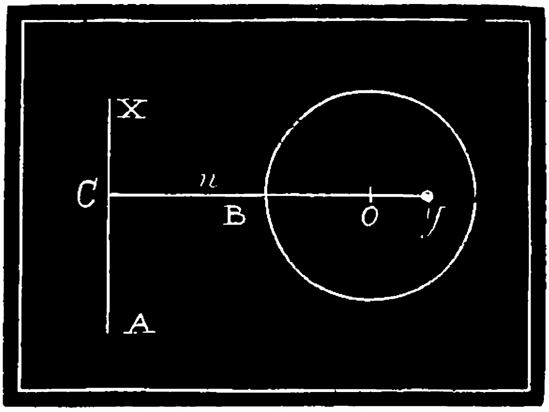
Fig. 43.



When a solid sphere, Fig. 44, revolving about an axis, A X, outside the body,

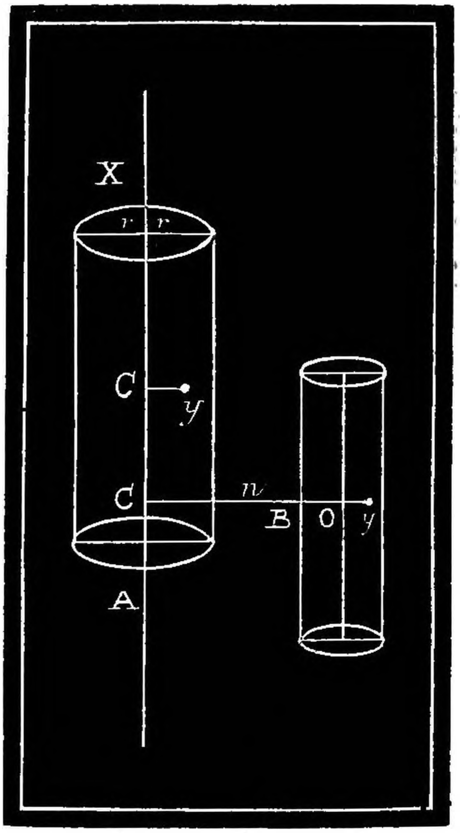
$$C\,y = \sqrt{n^2 + \tfrac{2}{5}\,r^2}$$
$$B\,O = r; \quad C\,O = n.$$

Fig. 44.



$Cy=r\sqrt{\frac{1}{2}}$; Fig. 45, a cylinder round its axis,

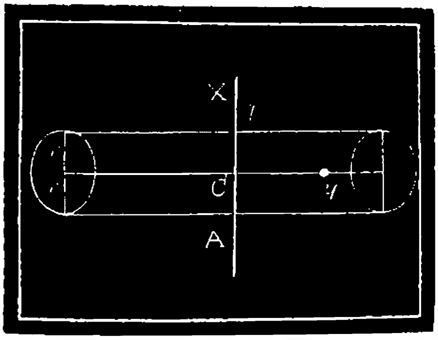
Fig. 45.


X
r
r
C
y
C
n
B
O
y
A

$C\,y=\sqrt{(n^2+\frac{1}{2}r^2)}$; $n=CO$, Fig. 45.

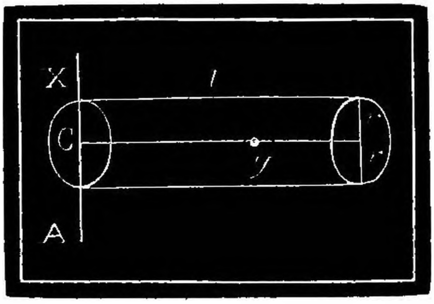
$$\mathrm{C}\,y=\sqrt{\left(\frac{l^2+3\,r^2}{12}\right)};$$ Fig. 46, l = the length.

Fig. 46.



$$\mathrm{C}\,y=\sqrt{\left(\frac{4\,l^2+3\,r^2}{12}\right)};$$ Fig. 47.

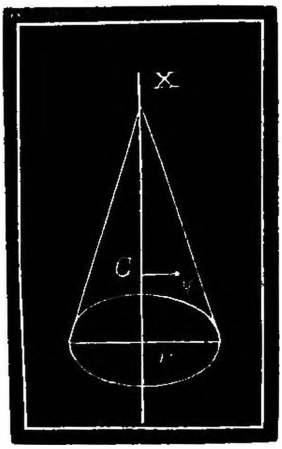
Fig. 47.



In a cone, Fig. 48, revolving about the axis,

$$\mathrm{C}\,y=r\,\sqrt{\tfrac{3}{10}}=\cdot 54772\,r.$$

Fig. 48.



The cone, Fig. 49,

$$C\,y = \sqrt{\left(\frac{2\,h^2 + 3\,r^2}{20}\right)}.$$

Fig. 49.

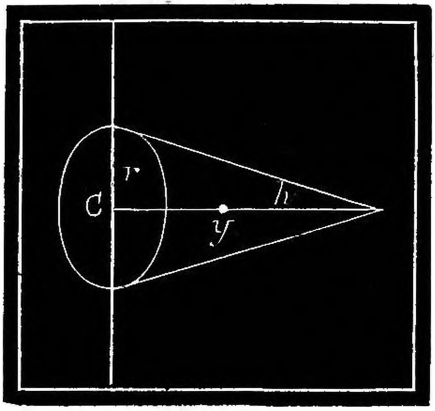
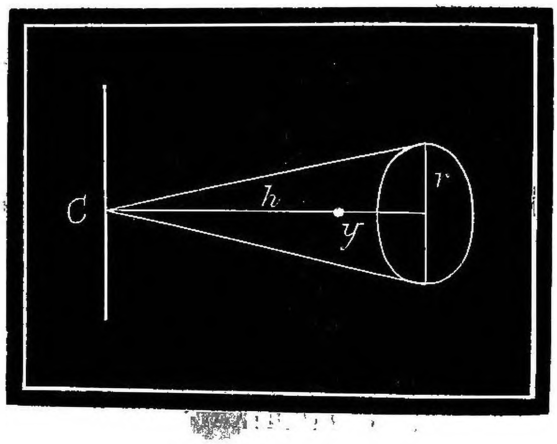
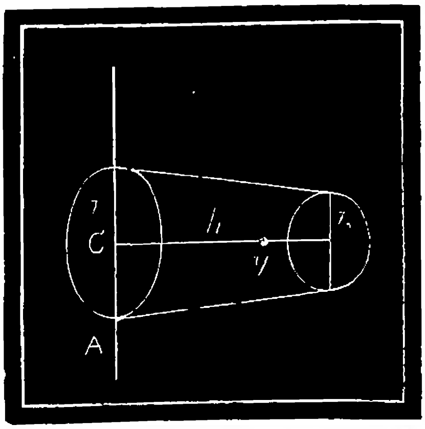


Fig. 50.



$$C\,y = \sqrt{\left(\frac{12\,h^2 + 3\,r^2}{20}\right)};\ \text{Fig. 50.}$$

Fig. 51.

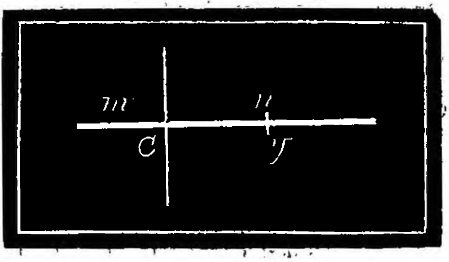


$$C\,y = \sqrt{\frac{h}{10}\left(\frac{r^2+3\,r\,r_1+r\,r_1^2}{r^2+r\,r_1+{}^1r_1^2}\right)+\frac{3}{20}\left(\frac{r^5-r_1^5}{r^3-r_1^3}\right)}$$

In a straight lever, whose arms are *m* and *n*, Fig. 52, the distance of the centre of *gyration*, *y*, from the axis A X, or,

$$C\,y = \sqrt{\left(\frac{m^3+n^3}{3\,m+3\,n}\right)}.$$

Fig. 52.



Parallelopiped, Fig. 53.

$$C\,y = \sqrt{\left(\frac{4\,l^2+b^2}{12}\right)};\quad E\,D = l,\ E\,F = b.$$

Fig. 53.

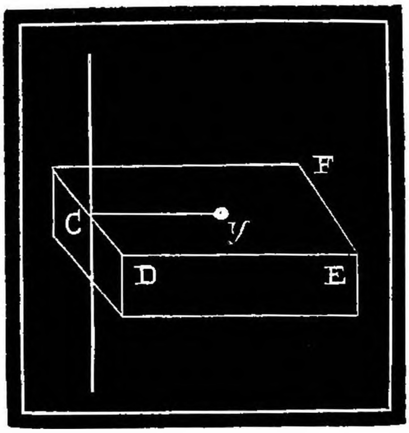


Fig. 54. Q P$=n$, D E$=l$

F E$=b$; E F$=b$

$$C\,y = \sqrt{\left(\frac{4\,l^2+b^2}{12}+n^2+a\,l\right)}.$$

Fig. 54.

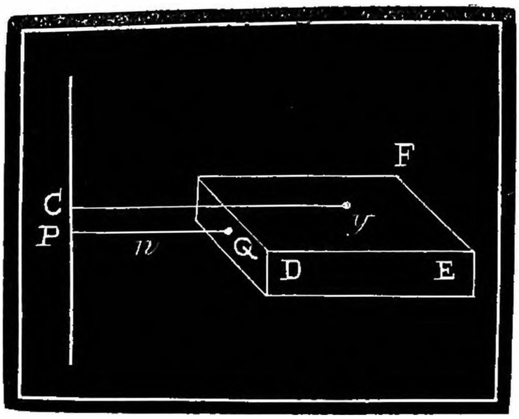
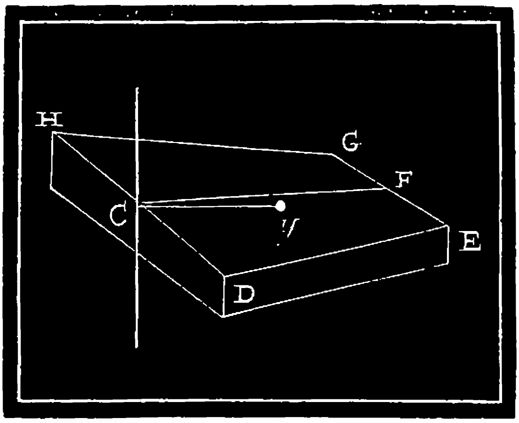


Fig. 55. $CF = l$; $HD = b$; $GE = b_1$.

$$Cy = \sqrt{\left(\frac{12\,l^2 + b^2 + b_1^2}{24}\right)}.$$

Fig. 55.

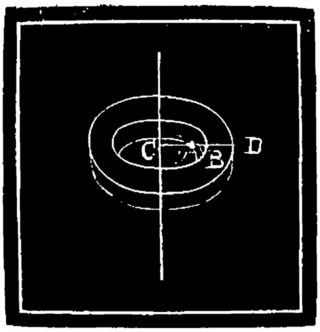


Cylindrical ring, Fig. 56,

$CB = r_1$; $CD = r$.

$$Cy = \sqrt{\left(\frac{r^2 + r_1^2}{2}\right)}.$$

Fig. 56.



In a spheroid of revolution, polar semi-axis a, equatorial radius r, turning on the polar axis,

$$Cy = r\sqrt{\tfrac{2}{5}}.$$

LIVING FORCE. VIS VIVA.

Facts go to show, and it seems reasonable to admit, that *forces are proportional to the degrees of velocity which they impress in equal very short times, upon the same body, yielding freely to their action, and in the proper direction of this action.*

This is taken as a fundamental axiom.

Then if two forces, which we call F and F_1, act upon the same body and impress it with or deprive it of the very small degrees of velocity, v and v_1 in the same element of time, t, then, according to the above axiom, we have, by the *rule of three :*

$$F : F_1 :: v : v_1 \; ; \quad (A).$$

To estimate and measure the force that we have designated by F, we compare it to another force whose effect on the given body is known—with gravity, for example—and as we know that the velocity, v_1, imparted to heavy bodies in an element of time, $t = g\,t$—that is (see pp. 92, 93),

$$v_1 = g\,t.$$

The value of g varies in different latitudes, and at different altitudes in the same latitude. In this work g is put$=32\frac{1}{6}$ a second.

The weight of the body, W, or the force exerted by gravity, may be put for F_1 in proportion (A), then (A) becomes

$$F : W :: v : g\,t = v_1$$

$$\therefore \quad F = \frac{W}{g} \; \frac{v}{t}; \qquad (B).$$

$\frac{W}{g} = M$ is constant quantity, for when g is changed W is changed in the same ratio. This constant ratio, M, is termed the *mass.* That M is a constant quantity may be thus shown : Let the same body, W, in any other latitude weigh W_1, and g changed to g_1; then, putting t for the same element of time in both places, we have

$$W : W_1 : : g\,t=(v) : g_1\,t=(v_1),$$

v, v_1 being very small degrees of velocity.

$$\therefore \quad \frac{W}{g}=\frac{W_1}{g_1}=M=\text{a constant quantity for all places.}$$

THE MEASURE OF MOTIVE FORCES AND INERTIA.

From equation (B) we can estimate the force called F in lbs. capable of imparting to or taking from a body of the weight W or mass M an element of velocity, v, in an element of time, t, as

$$F=\frac{W}{g}\,\frac{v}{t}=M\,\frac{v}{t}. \quad \text{(C)}.$$

When the ratio $\frac{v}{t}$ is constant, which is the case with uniformly accelerated or retarded motion, F is a constant force. But, since to communicate to a body of the weight W a variation of velocity, v_1, in an element of time, t, there must be developed an effort, $\frac{W}{g}\,\frac{v}{t}$; then there is a resistance to be overcome, of which this effort is the measure. This resistance is the force of inertia, the reaction which takes place every time that a variation of motion is produced. Thus equation (C) gives at one and the same time the measure of the motive force which produces the change of motion, and that of the force by which the body, by virtue of its inertia, opposes this change. An examination of equation (C) will show that, for a weight, W, or a given *mass*, M, the magnitude of the force, F, will increase as the change of motion becomes more rapid, or as the ratio $\frac{v}{t}$ becomes greater. It is thus we account for the magnitude of efforts and reactions developed in the transmission of motion by the shocks experienced between hard bodies, in a very short interval of time, when the velocity varies or is destroyed suddenly. An example will put this matter in a very clear light. In case we require the quantity of motion imparted to a ball weighing 38·6 lbs., and upon which gunpowder has impressed a velocity of 1600 feet a second,

$$F\,t = M\,v.$$

$$\text{The } mass, \ M = \frac{38{\cdot}6}{32\frac{1}{6}} = 1{\cdot}2\,; \quad \therefore \ F\,t = 1920.$$

If we suppose successively,

$$t = 1{\cdot}00'', \quad 0{\cdot}50'', \quad 0{\cdot}10'', \quad 0{\cdot}01'',$$

we have,

$$F = 1920 \text{ lbs.}, \ 3840 \text{ lbs.}, \ 19200 \text{ lbs.}, \ 192000 \text{ lbs.}$$

When a body has only a motion of translation, the *vis viva*$=$(*mass*) × (velocity)$^2 = \frac{W}{g}\,v^2$; hence the units of work conservated in a body so moving=half the *vis viva*$= \frac{v^2\,W}{2g}$ —a result before established on the principle of work (p. 97). When a body has a motion of rotation and a motion of translation, the *vis viva*=*vis viva* due to translation+*vis viva* due to rotation. *Vis via* of rotation=(*mass*) (angular velocity)$^2 = \frac{W}{g}$ × (angular velocity)^2.

Let the body A=5 lbs., moving with a velocity of 8 ft. a second; B=9 lbs., moving with a velocity of 11 ft. a second; required the accumulated work in these bodies in terms of what writers on mechanics call *vis viva.*

$$\text{Work in A} = \frac{8^2 \times 5}{2 \times 32\frac{1}{6}}\,; \quad \text{Work in B} = \frac{11^2 \times 9}{2 \times 32\frac{1}{6}}$$

$\therefore$ The units of work in A and B together=

$$\tfrac{1}{2}\,\frac{8^2 \times 5 + 11^2 \times 9}{32\frac{1}{6}} =$$

$\therefore$ The units of work accumulated$=\frac{1}{2}$ the *vis viva.*

Ques. 27. What must be the weight of a single body, D, moving with the velocity of 10 feet a second, so that its accumulated work may be equal to that of the bodies A, B, C, weighing 4, 5, 6 lbs. respectively, and moving, with a velocity of 9, 8, 7 feet a second respectively?

Let W=the weight of the body D, then the units of work in D=

$\frac{10^2 \times W}{64\frac{1}{3}}$, which has to be made equal to the units of work in A, B, and C together=

$$\frac{9^2 \times 4}{64\frac{1}{3}}+\frac{8^2 \times 5}{64\frac{1}{3}}+\frac{7^2 \times 6}{64\frac{1}{3}}.$$

$$\therefore \quad 10^2 \times W=9^2 \times 4+8^2 \times 5+7^2 \times 6; \quad (R).$$

$$\therefore \quad W=9{\cdot}38.$$

It is evident that if the bodies A, B, C, D be parts of the same machine, when the velocity of one of these bodies is increased, say n times, the velocities of the other bodies are also increased n times, and we establish the important fact that the weight W has not to be altered in the same machine. On this supposition, equation (R) becomes

$$(10n)^2 \times W=(9n)^2 \times 4+(8n)^2 \times 5+(7n)^2 \times 6,$$

which, when divided by n^2, becomes equation (R), and W=9·38 lbs., as before.

THE MAXIMUM VELOCITY OF THE PISTON OF A STEAM ENGINE.

The maximum velocity will evidently take place at a point in the stroke where the pressure of the steam in the cylinder is equal to the whole pressures of the resistances acting against the steam; for, so long as the pressure of the steam is greater than the resistances, the motion of the piston becomes accelerated; at the point indicated, the work in the piston will be equal to the work accumulated in the machine; from this property the velocity of the piston may be found.

Ques. 28. Given the pressure of the steam = 45 lbs. and the gross load on the piston = 18·5 lbs. on the square inch; at what point of the stroke will the velocity be greatest, supposing the steam cut off at $2\frac{1}{2}$ feet of the stroke?

Let x be put = the required length of stroke; then, according to Boyle's law, the pressure at this point in lbs.

$$=\frac{2\frac{1}{2}\times45}{x}.$$

But we also know that the velocity is greatest when the pressure = 18·5 lbs. on the square inch.

$$\therefore\quad 18{\cdot}5=\frac{2\frac{1}{2}\times45}{x}.$$

$$\therefore\quad x=6\tfrac{3}{37}\text{ feet.}$$

Ques. 29. The length of the stroke in a condensing engine is 12·3 feet, the pressure of the steam 43·7 lbs., and the steam cut off 3·7 feet of the stroke. Find the gross load upon each square inch, and the point at which the velocity of the piston is greatest?

$$\frac{12{\cdot}3}{3{\cdot}7}=3{\cdot}324327.$$

The dual logarithm of 3·324327 may be calculated in a minute or two; it is = 120126729, which, when divided by 10^8, = 1·20126729. In the language of dual arithmetic, this operation is concisely expressed thus:

$$\frac{\downarrow,\left(\dfrac{12{\cdot}3}{3{\cdot}7}\right)}{10^8}=1{\cdot}20126729.$$

The amount of work on one square inch of piston in a single stroke =

$$\mathbf{43{\cdot}7\times3{\cdot}7\times1{\cdot}2013+43{\cdot}7\times3{\cdot}7=43{\cdot}7\times3{\cdot}7}$$
$$\mathbf{\times(2{\cdot}2013)=355{\cdot}928197}.$$

The mean pressure on the square inch =

$$\frac{355{\cdot}928197}{12{\cdot}3}=28{\cdot}94\text{ lbs.}$$

It is evident that the velocity is greatest at that point of the stroke where the expended steam attains the pressure of 28·94 lbs. Then, putting x for this point in the stroke, we have, by Boyle's law,

$$28{\cdot}94 : 43{\cdot}7 :: 3{\cdot}7 : x$$
$$\therefore \quad x = 5{\cdot}59.$$

It is further evident that the length of the stroke, 12·3 divided by 2·20126729, gives the points x also.

Ques. 30. The length of stroke = 9·76 feet, pressure of steam = 45·3 lbs., steam cut off at 2·5 feet of the stroke, the area of the piston = 4185·9 square inches, the weight moved having the same motion as the piston = 72640 lbs.; what will be the maximum velocity of the piston?

$$\frac{9{\cdot}76}{2{\cdot}5} = 3{\cdot}904.$$

The dual logarithm of 3·904 divided by $10^8 = 1{\cdot}37200166$; then, according to what has been shown in the last example, $\dfrac{9{\cdot}76}{2{\cdot}37200166} = 4{\cdot}115$ feet, the length of stroke made when the velocity is greatest.

The work of the steam up to this point, on the whole piston, =

$$45{\cdot}3 \times 2{\cdot}5 \times (1{\cdot}49834708) \times 4185{\cdot}39 \times 710210.$$

Since the dual logarithim of $\dfrac{4{\cdot}115}{2{\cdot}5} = ↓, (1{\cdot}646)$ divided by 10^8 $= {\cdot}49834708$, to which 1 is added.

The units of work by the whole piston in a single stroke =

$$45{\cdot}3 \times 2{\cdot}5\ (2{\cdot}372) \times 4185{\cdot}39 = 1124317.$$

For $2{\cdot}372 = ↓, (3{\cdot}904)$ divided by $10^8 + 1$.

$\therefore$ Mean pressure on the square inch $= \dfrac{45{\cdot}3 \times 2{\cdot}5 \times 2{\cdot}372}{9{\cdot}76} =$ 27·52 lbs.

But the work done upon the resistances up to the point (4·115) of the stroke where the velocity is a maximum =

$$4{\cdot}115 \times 27{\cdot}52 \times 4185{\cdot}39 = 473974$$

∴ The accumulated work in the piston when the velocity is a maximum =

$$710210-473974=236236.$$

Then, putting v for the maximum velocity of the piston in feet a second,

$$\frac{v^2 \times 72640}{64\frac{1}{3}}=236236$$

$$\therefore \quad v=14{\cdot}464 \text{ feet.}$$

Ques. 31. Taking the data of the last example, find the weight (W) of the mass moved, supposed to be posited in a position to have the same motion as the piston when its maximum velocity is 5 feet a second?

W may be found from knowing that

$$\frac{5^2 \times W}{64\frac{1}{3}}=236236.$$

$$\therefore \quad W=639713 \text{ lbs.}=286 \text{ tons nearly.}$$

Ques. 32. The length of the stroke in a condensing engine=10 ft., the pressure of the steam=30 lbs., the steam cut off at 2 ft. of the stroke, the area of the piston 4000 square inches, and the weight of the mass moved having the same motion as the piston=50000 lbs.; what is the velocity of the piston when 4 ft. of the stroke is made?

Total work on one square inch of the piston =

$$30 \times 2 \times (2{\cdot}60943791)=156{\cdot}566.$$

For the dual logarithm of $\frac{10}{2}$ =160943791, and 1. added 160943791, divided by 10^8, gives 2·60943791.

The mean pressure of the steam, or gross load, $=\frac{156{\cdot}566}{10}=$ 15·657 lbs.

The work done on the piston when 4 feet of the stroke is made=

$$4000 \times 30 \times 2 \times (1{\cdot}69314718)=406355.$$

Since the dual logarithm of $\frac{4}{2}=2,=69314718.$

But the excess of this work over the work expended in moving the resistance shows the units of work accumulated in the piston when 4 ft. of the stroke is made. Work of resistance up to this point of the stroke=

$$15{\cdot}657 \times 4000 \times 4 = 250512.$$

∴ Work accumulated in the piston=

$$406355 - 250512 = 155843.$$

v being put for the velocity, then

$$\frac{v^2 \times 50000}{64\frac{1}{3}} = 155843.$$

$$\therefore \quad v = 14{\cdot}16 \text{ ft.}$$

In the same manner the velocity may be found for any part of the stroke. The weight of the mass moved in any engine, referred to the piston, may be found as in example 23. The mechanical principles evolved in the last four examples are worthy of particular attention.

Before the author of the present work, Oliver Byrne, discovered the art and science of dual arithmetic, independent and direct solutions of such questions as the following defied the combined skill of mathematicians. In such cases results were generally guessed at, or roughly approximated to, by the help of empirical rules and limited tables. Questions that presented great difficulties may now be solved with ease, and without extraneous aids or methods of approximation; the subject is merely touched upon in this work to call forth, in the young student, a spirit of inquiry respecting an art so easily acquired, and at the same time extensive in its application.

Ques. 33. Steam of 60 lbs. pressure (p) is cut of at x feet of the stroke, and expands, according to Boyle's law, so that the mean pressure throughout the stroke=32 lbs. (q), the whole stroke=7·5 ft. (a); find x by a direct process, without the use of tables or empirical rules?

We have before shown that $\downarrow, \left(\frac{a}{x}\right)$ is read the dual logarithm of $\frac{a}{x}$, which any student may calculate by common addition and subtraction.

$$\text{Then } \frac{p\,x\left(\frac{\downarrow,\left(\frac{a}{x}\right)}{10^8}+1\right)}{a}=q.$$

$$\therefore\quad x\downarrow,\left(\frac{a}{x}\right)+10^8 x=10^8\frac{a\,q}{p}$$

$$\therefore\quad x\downarrow, a,-x\downarrow,(x)+10^8 x=10^8\frac{a\,q}{p}$$

$$\therefore\quad x\,(\downarrow, a,+10^8)-10^8\frac{a\,q}{p}=x\downarrow,(x).$$

Putting M for $10^8\frac{a\,q}{p}$, and N for $\downarrow, a+10^8$; the last equation may become

$$x\,\mathrm{N}-\mathrm{M}=x\downarrow,(x).$$

Then a natural number n may be found of which N is the dual logarithm; and a natural number, m, of which M is the dual logarithm.

$$\therefore\quad \frac{n^x}{m}=x^x.$$

$$\therefore\quad \frac{1}{m}=\frac{x^x}{n^x}=\left(\frac{x}{n}\right)^x.$$

Taking the $\frac{1}{n}$ root of both sides of the last equation, and substituting z for $\frac{x}{n}$, then,

$$\left(\frac{1}{m}\right)^{\frac{1}{n}}=\left(\frac{x}{n}\right)^{\frac{x}{n}}=z^z.$$

$$10^8\,\frac{7\cdot5\times32}{60}=400000000,=\downarrow,(m).$$

$$\therefore\quad \downarrow,\left(\frac{1}{m}\right)={}'400000000;\ \downarrow,(7\cdot5)=201490302.$$

$$\therefore\quad \downarrow,(7\cdot5)+10^8=\downarrow,(n)=301490302,\ \downarrow,\left(\frac{1}{n}\right)= {}'301490302$$

$$\therefore\quad \frac{1}{n}=\cdot049050593$$

$$\text{Then} \downarrow, \left(\frac{1}{m}\right)^{\frac{1}{n}} = z \downarrow, (z) = {}'400000000\ (\cdot 049050593) =$$
$$'19622037.$$

The general solution of equations of the form

$$z \downarrow, (z) = {}'19620237$$

is given in Dual Arithmetic a New Art, Part II., page 98. By this new art z is readily found to be = the dual number $\frac{1}{10} \downarrow$ '34,5, '2'4'6'6'0, which is equal to the natural number ·076221291.

$$z = \frac{x}{n} = (\cdot 049050598)\ x = \cdot 076221291.$$

$$\therefore \quad x = 1{\cdot}53932 \text{ feet.}$$

VELOCITY ACQUIRED BY BODIES IN DESCENDING AN INCLINED PLANE OR CURVE.

When a body descends an inclined plane by the force of gravity, the work accumulated in the body is that which is due to the perpendicular elevation, without regard to the angle or curvature of the plane. Therefore, the velocity of the body will be that which it would acquire by falling freely along the perpendicular height of the curve or plane.

Ques. 34. What velocity will a body acquire in descending an inclined plane, whose perpendicular height is 579 ft.?

Putting t for the time of descent in seconds,

$$t^2 \times 16\tfrac{1}{12} = 579.$$
$$\therefore \quad t^2 = 36, \text{ or } t = 6 \text{ seconds.}$$

Consequently, the velocity acquired at the end will be= $6 \times 32\frac{1}{6} = 196$ ft. a second.

Ques. 35. A train of 50 tons descends an incline of 1200 feet, having a total rise of 170 feet; find the velocity acquired by the train supposing the friction to be $6\frac{1}{2}$ lbs. a ton?

Work done by gravity =

$$50\times2240\times170=19040000.$$

Work due to friction =

$$6\tfrac{1}{2}\times50\times1200=390000.$$

Then the units of work in the train at the bottom of the incline will be

$$19040000-390000=18650000.$$

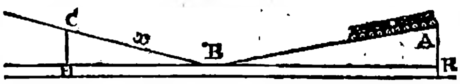
Putting v for the required velocity, then 50 tons $=112000$ lbs., and

$$\frac{v^2\times112000}{64\frac{1}{3}}=18650000.$$

$$\therefore\quad v=103\cdot5 \text{ feet a second.}$$

Ques. 36. A train of 20 tons descends an incline, A B, Fig. 57, of 360 feet, having a total rise, A R, of 6 feet, and ascends another incline, B C, having a rise of 1 foot in 20; how far will the train ascend the plane B C before returning to B, and what will be the velocity of the train on its return to B, supposing the friction to be 8 lbs. a ton on A B, and 6 lbs. on B C?

Fig. 57.



20 tons $=44800$ lbs. Units of work accumulated in descending from A to B =

$$44800\times6-20\times8\times360=211200.$$

Then, putting x for the length of the plane, B C, to which the train will ascend, then $\frac{x}{20}=$ H C, the height; therefore,

$$44800\times\frac{x}{20}+20\times6\times x=211200.$$

$$\therefore\quad x=89\,\frac{29}{59}\ \text{and}\ \frac{x}{20}=4\,\frac{28}{59}.$$

On returning to B, the units of work accumulated in the train will be

$$44800\times4\,\frac{28}{59}-20\times6\times89\,\frac{29}{59}=189422\,\frac{2}{59}.$$

Putting v for the velocity at B, then

$$\frac{v^2 \times 44800}{64\frac{1}{3}} = 18942^2\tfrac{2}{9}$$

$$\therefore \quad v = 16{\cdot}49 \text{ ft. a second.}$$

Ques. 37. The weight of a fly-wheel is 2000 lbs., the centre of gyration describes a circle 40 feet in circumference, the circumference of the axis upon which it turns is 10 inches, the wheel when stopped was making 30 revolutions a minute, but came to a state of rest in 48 revolutions; what part of the weight of the wheel is the friction upon the axis?

Units of work in the wheel=

$$\frac{20^2 \times 2000}{64\frac{1}{3}} = 12435{\cdot}23.$$

Since the velocity of the centre of gyration $= 20$ feet $= \dfrac{40 \times 30}{60}$. The circumference of the axis $= \frac{10}{12} = \frac{5}{6}$ feet. The length of the path in contact during 48 revolutions $= \frac{5}{6} \times 48 = 40$ feet. Putting f for the coefficient,

$$f \times 2000 \times 40 = 12435.$$

$$\therefore \quad f = \tfrac{5}{32} \text{ nearly.}$$

Since a cubic foot of shaken gunpowder weighs about 960, and a cubic foot of cast iron 7200 ounces, it is readily shown that if x = diameter of calibre of a cannon in inches, a charge of gunpowder $\frac{1}{3}$ the weight of the round shot will fill $\dfrac{5x}{3}$ inches length of bore. For, let y = length of bore filled with the gunpowder, then π being put $= 3{\cdot}14159265$,

$$\frac{\pi x^2}{4} \times y \times \frac{960}{1728} = \text{ounces of gunpowder,}$$

$$\text{and} \quad \frac{\pi x^3}{6} \times \frac{7200}{1728} = \text{ounces weight of round shot;}$$

$$\therefore \quad 3 \times \frac{\pi x^2}{4} \times y \times \frac{960}{1728} = \frac{\pi x^3}{6} \times \frac{7200}{1728}.$$

$$\therefore \quad y = \frac{5x}{3} \text{ inches} = \tfrac{5}{36}\, x \text{ feet,}$$

generally $y = n\,x$, $\quad n = \frac{5}{36}$.

Ques. 38. Suppose the length of the bore of a cast-iron cannon $=12x$ inches $(m\,x)$, diameter of bore $=x$ inches, length occupied by the powder $=\frac{5x}{3}$ inches $(n\,x)$; the powder being exploded expands, according to Boyle's law, from the space first occupied to fill the whole bore, the force thus developed gives an initial velocity of v feet (1200) a second to a round-shot x inches diameter. Required the pressure (p) of the gas on the square inch before expansion takes place?

The pressure of the atmosphere on the shot as it passes along the bore is in a great measure counteracted by the gas which escapes around and before the shot.

$$12\,x \text{ divided by } \frac{5\,x}{3}=\frac{36}{5}=\frac{m}{n};$$

The dual logarithm of $\frac{36}{5}$ $=197408103$,

$$\frac{197408103}{10^8}=1{\cdot}97408103.$$

Put $\phi=$ the dual logarithm of $\frac{m}{n}$ divided by 10^8.

$\frac{5\,x}{36}\,p\,\phi=n\,x\,p\,\phi=$ units of work on each square inch of circular section of calibre by the expansion of the powder. Therefore the units of work done on the round-shot in passing the length of the bore $=n\,x\,p\,\phi\times\frac{\pi\,x^2}{4}=\frac{n}{4}\pi\,x^3\,p\,\phi$.

But the units of work in the shot is also equal to its weight in pounds multiplied by $\frac{v^2}{2\,g}$.

g being put for $32\frac{1}{6}$.

Weight of shot $=\frac{25}{36}\pi\,x^3$ ounces $=\frac{25}{16\times 36}\pi\,x^3$ lbs. $=c\,\pi\,x^3$, if c be put for the specific gravity of the shot divided by 1728 $\times 16\times 6$.

$\therefore \quad c\,\pi\,x^3 \times \frac{v^2}{2g}$ is also = the units of work in the shot.

$$\therefore \quad \frac{n}{4}\pi x^3 p\,\phi = c\,\pi\,x^3\,\frac{v^2}{2g}$$

$\therefore \quad \frac{n}{2}p\,\phi = c\,\frac{v^2}{g}$, an expression independent of both π and x.

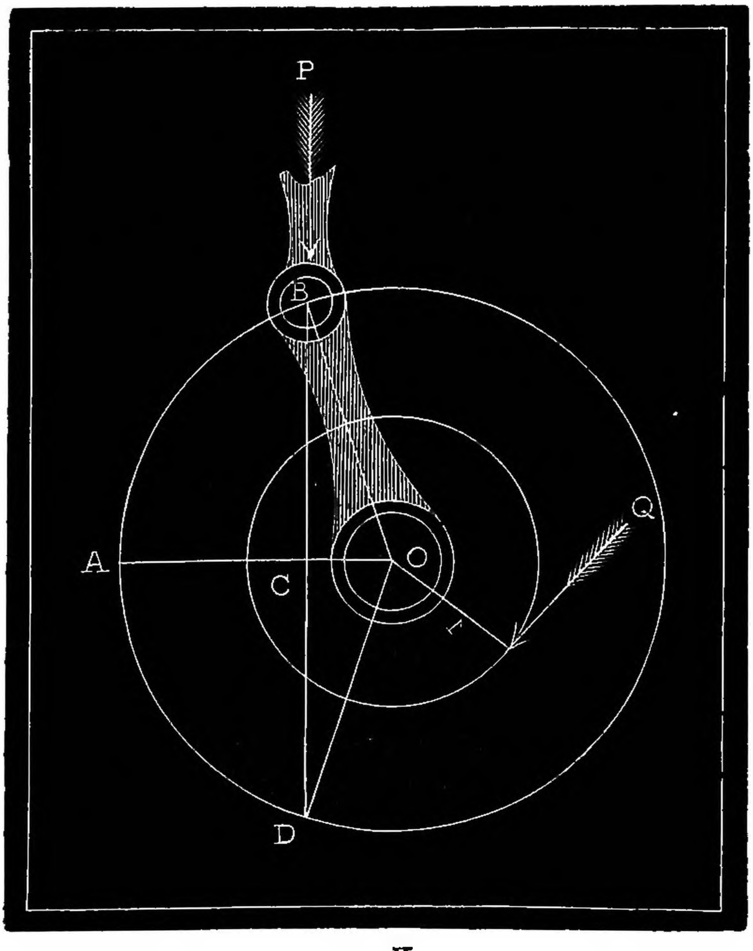
$$p = \frac{2\,c\,v^2}{n\,g\,\phi} = 14173\cdot3164 \text{ lbs.}$$

PROBLEM.

To find the position of the crank corresponding to its maximum and minimum velocity in a single-acting engine.

Let O B and O D be the required positions on the crank, and let P represent the constant pressure of the con-

Fig. 58.



necting-rod, supposed to act in a vertical line. Put Q = the constant resistance, acting at one foot from the axis of the fly-wheel, equivalent to the work of the engine.

The motion will be accelerated from B to D. This acceleration will commence when the moving pressure is equal to the resisting pressure, and it will cease under the same condition. The former will correspond to the position of minimum, the latter to that of maximum velocity. Hence, at these two points the moment of P must be equal to the moment of Q, and the point D will be as much below the horizontal line A O as the point B is above it.

$$\therefore \quad P \times CO = Q \times 1. \qquad (1).$$

Again, by the equality of work, putting

$$r = OB,$$

Units of work by P in 1 revolution $= 2r\,P$.

" " Q " $= Q \times 2 \times 3{\cdot}1416$

$$\therefore \quad 2r\,P = Q \times 2 \times 3{\cdot}1416 \qquad (2).$$

Dividing equation (1) by equation (2),

$$\frac{CO}{r} = \frac{1}{3{\cdot}1416} = {\cdot}31831$$

∴ From the table of natural sines

$${\cdot}31831 = \text{cosine of } 71^\circ\ 27';$$

for $\frac{CO}{r}$ is the cosine of the angle B O C.

PROBLEM.

To find the dimensions of the fly-wheel, such that its angular velocity may at no point differ from the mean velocity beyond a certain limit.

Let d and p be the maximum and minimum velocities of the wheel at the distance of one foot from the axis; W the weight of the wheel, and k the distance of the centre of gyration from the axis.

$$\text{Work of P from B to D} = P \times BD =$$
$$P \times 2r \times \sin 71^\circ\ 27' =$$
$$2r\,P \times {\cdot}948.$$

Work of the constant pressure Q from D to B =

$$\frac{\mathbf{Q}\times\mathbf{2}\times\mathbf{3{\cdot}1416}\times\mathbf{142°54'}}{\mathbf{360°}}=$$

$$\mathbf{2}\,r\,\mathbf{P}\times\mathbf{{\cdot}3968},$$

by putting for $\mathbf{Q}\times\mathbf{2}\times\mathbf{3{\cdot}1416}$ its value $\mathbf{2}\,r\,\mathbf{P}$, found in the last problem.

Now the difference of these will give the *work* that goes to increase the speed of the wheel between the points B and D, that is, *work* going into the wheel between B and D =

$$\mathbf{2}r\,\mathbf{P}\times\mathbf{{\cdot}948}-\mathbf{2}\,r\,\mathbf{P}\times\mathbf{{\cdot}3968}=r\,\mathbf{P}\times\mathbf{1{\cdot}1022}.$$

Accumulated work at $\mathrm{B}=\frac{p^2\,k^2\,\mathbf{W}}{\mathbf{2}\,g}.$

Accumulated work at $\mathrm{D}=\frac{d^2\,k^2\,\mathbf{W}}{\mathbf{2}\,g}.$

Hence the accumulated work gained from B to D=

$\frac{k^2\,\mathbf{W}}{\mathbf{2}\,g}\;(d^2-p^2)$, but this must be equal to the work before found;

$$\therefore\quad \frac{k^2\,\mathbf{W}}{\mathbf{2}\,g}\;(d^2-p^2)=r\,\mathbf{P}\times\mathbf{1{\cdot}1022}\qquad(3).$$

Let V be the mean velocity of the wheel at one foot from the axis, and let the extreme velocities d and p differ from this mean velocity by the nth part; then

$$d=\mathrm{V}+\frac{\mathrm{V}}{n}\text{ and }p=\mathrm{V}-\frac{\mathrm{V}}{n};$$

$$\therefore\quad d^2-p^2=\frac{4\mathrm{V}^2}{n}\qquad(4).$$

Let N be the number of double strokes performed a minute, then

$$\mathrm{V}=\frac{2\times3{\cdot}1416\times\mathrm{N}}{60}={\cdot}10472\times\mathrm{N};\qquad(6).$$

Let $\mathbf{U}$ be the work of the engine, then

$$\mathbf{U}=2\,r\,\mathbf{P}\,\mathbf{N}\quad\therefore\quad r\,\mathrm{P}=\frac{\mathrm{U}}{2\,\mathrm{N}}\qquad(5).$$

Substituting the values given in the equations (4), (5), (6), in equation (3),

and reducing, $2g=32\frac{1}{6}$

$$W=\frac{n\,U}{k^2\,N^3}\times\frac{32\frac{1}{6}\times 1{\cdot}1022}{4\times({\cdot}10472)^2}$$

$$\therefore\quad W=\frac{n\,U}{k^2\,N^3}\times 808{\cdot}2 \qquad (7),$$

which is the expression for the weight of the fly-wheel in pounds.

If H be put for the horse power of the engine, then $U=33000\,H$; substituting this in equation (7), and reducing to tons,

$$W=\frac{n\,H}{k^2\,N^3}\times 11907 \qquad (8),$$

which is the expression in units of tons. Let R = the mean radius of the fly-wheel, e = depth of the rim, then from a well-known property of the centre of gyration

$$k^2=R^2+\frac{e^2}{4};$$

substituting this in equation (8), then

$$W=\frac{n\,H}{\left(R^2+\frac{e^2}{4}\right)N^3}\times 11907.$$

Neglecting $\frac{e^2}{4}$ as being comparatively small, then

$$W=\frac{n\,H}{R^2\,N^3}\times 11907 \qquad (9).$$

It may be observed that the weight of the wheel varies inversely as the cube of the number of strokes performed by the engine per minute.

If a=the area of the section of the rim in square feet, and 450 lbs. be taken as the weight of a cubic foot of the metal, then $W=2\pi\,R\,a\,\frac{450}{2240}$ tons nearly. Substituting in equation (9), and solving the resulting equation for R,

$$\text{then } R=\left(\frac{n\,H}{N^3\,a}\times\frac{11907\times 2240}{2\times 3{\cdot}1416\times 450}\right)^{\frac{1}{3}}$$

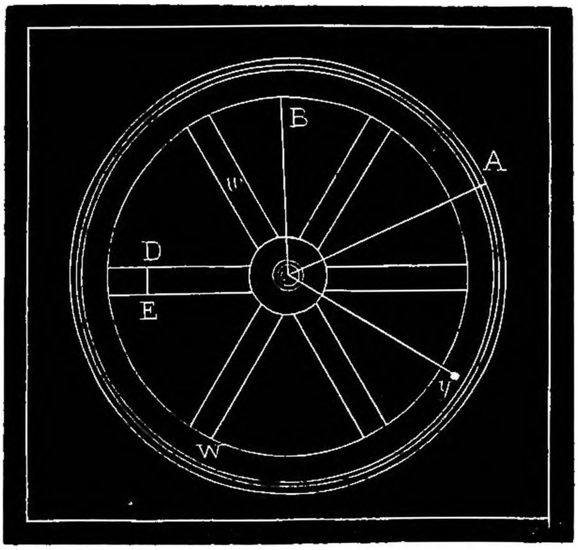
$$\therefore \frac{21}{N}\sqrt[3]{\frac{n\,H}{a}}=R,$$

which is an expression for the mean radius of a cast-iron fly-wheel of a single-acting engine, when there are given the number of strokes of the piston, the horse power, the area of the mean section of the rim, and the proportional variation from a mean velocity. Proceeding in the same way, for the double-acting engine,

$$\text{cosine B O A}=2\times{\cdot}31831,$$

$$\text{and } R=\frac{12}{N}\left(\frac{n\,H}{a}\right)^{\frac{1}{3}}$$

Fig. 59.



In the fly-wheel B A W, Fig. 59, C A=R; C B=r, the outer and inner radius of the ring. W=weight of the ring in pounds; w the weight of the arms, breadth of each, D E=b. If y be the centre of gyration of the ring and arms, then, putting y=C y,

$$y=\sqrt{\frac{6\ W\ (R^2\times r^2)+w\ (4\ r^2\times b^2)}{12\ (W+w)}}$$

In practice, the centre of gyration, including the ring and arms, may be assumed at $y=r$ the length of the inner radius from the centre, C. Putting a for the angular velocity or number of revolutions a minute, at the end of the time t in which the fly-wheel would concentrate the same power as the steam engine, t may be taken$=128$ seconds; but when the work is irregular, t may be taken as high as 170 seconds. Taking these average quantities, the weight of a fly-wheel for a given horse power H will be

$$\text{Weight}=\frac{(11)^3\times(10)^2\times128\text{ H}}{a^2\ r^3}.$$

Ques. 39. Required the weight of a fly-wheel when the engine is of 56 horse power; the inner radius of the ring$=$ 10 ft. making 42 revolutions a minute?

$$\text{Weight}=\frac{133100\times128\times56}{42^2\times10^2}=5409\text{ lbs. nearly.}$$

IMPACT.

It has been before shown that the *mass* (m) of a body is a constant quantity at all heights and in all latitudes, while the weight (w) and the value of g are variable; but $m=\frac{w}{g}$ under all circumstances.

There is much uncertainty and error involved in the methods employed by philosophers to find the value of g in different places. In this work, for the want of knowing better, g is put$=32\frac{1}{6}$ ft. That is, a body falling from a state of rest is supposed to be moving at the end of the first second with a velocity of $32\frac{1}{6}$ ft. a second. When we say absolutely and without other explanation that the quantity g, which expresses the acceleration produced by gravity, is the measure of this force, we give to students an incorrect idea, since g is in reality only the velocity imported to or taken from a body by gravity during each second of its action, and the velocity which is expressed in feet cannot measure a force which should be compared with pounds.

MOMENTUM OR QUANTITY OF MOTION.

The products $m\ v$, $m\ V$, equal to $\frac{w}{g}v$ or $\frac{w}{g}V$, have received the name *momentum;* it is a conventional phrase, to which we attach no other signification than that of the product of the *mass* into the velocity imparted to or taken from it.

If $w = 193$ lbs. where $g = 32\frac{1}{6}$ ft., then the mass will be $= \frac{193}{32\frac{1}{6}} = 6$. In Paris g is said to be $= 32{\cdot}1817$ ft., in which place w would be $= 193{\cdot}0902$ lbs.; but the mass remains unaltered, for $\frac{192{\cdot}0902}{32{\cdot}1817} = 6$ also.

It may be further observed that the product $m\ v$, $m\ V$, is equal to $f\ t$ or $f\ T$ of the force, and time during which it has acted. If we consider two forces as acting for different times upon two bodies of unequal masses, we shall have $f\ t = m\ v$, and $F\ T = M\ V$;

$$\text{and} \therefore\ f\,t \ : \ F\,T \ :: \ m\,v \ : \ M\,V,$$

whence it follows that the quantities of motion $m\ v$, $M\ V$, imparted to or taken from different bodies in unequal times, are as the product of the forces to which they are due into the time during which these forces have acted. It is only when the times are equal that the quantities of motion impressed or destroyed are proportional to the forces, and can serve for their measure. From these remarks it follows, as we shall presently explain, that in shocks there is no loss of quantity of motion, which is expressed in saying that there is a preservation of the quantities of motion. But we shall show hereafter that shocks occasion a loss of work.

If the forces are equal and act during the same time, the quantities of motion imparted or destroyed, in the two bodies with masses m and M, are equal. This occurs in the reaction of two bodies which press, push, or impinge upon each other. The effects of compression and resistance being equal, opposed and developed during the same time, it follows that the quantity of motion imparted, in the reaction, to one of the bodies is equal to that which is lost by the other. This is a fact which is a necessary consequence of the theory of the shocks of bodies.

Thus, for example, when a body with a mass m impressed with a velocity v, impinges on a body with a mass M animated with a velocity V, in the same line, whether in the same or opposite directions, it develops at the point of contact equal and opposite efforts of compression, in an element of time t, taking from the impinging body a small degree of velocity v_1 and consequently a quantity of motion $m\,v_1$, and imparting to the body shocked, if it moves in the same direction as the first, an increase of velocity V_1, and a quantity of motion $M\,V_1$. These quantities being equal, we have then, at each instant of the mutual shock or compression of bodies, $m\,v_1 = M\,V_1$. In this case one of the bodies loses a quantity of motion equal to that gained by the other, and the sum of their two quantities of motion remains the same. The same thing transpiring at each instant of the shock, it follows then that the total quantity of motion lost by a body is equal to that gained by the other during the compression, and that at each end of this period, the sum of the quantities of motion is the same after the shock as before. This consequence constitutes the principle of the

CONSERVATION OF THE QUANTITIES OF MOTION;

otherwise termed,

THE PRINCIPLE OF THE CONSERVATION OF MOTION OF THE CENTRE OF GRAVITY.

When operating with soft bodies, whose elasticity is completely impaired by the shock, and which after compression unite and travel together with a common velocity u, the quantity of motion after the shock is $(m+M)\,u$, and from what proceeds we should have

$$m\,v + M\,V = (m+M)\,u.$$

∴ We derive for the common velocity after the shock

$$u = \frac{m\,v + M\,V}{m+M} \qquad \text{(I.)}$$

Ques. 40. Let a body, A, weighing 772 lbs. move with a velocity of 7 ft. a second, and impinge on a body, B, weighing 965 lbs., moving the same way in a line joining their centres of gravity with a velocity of 4 ft. a second, and after compression suppose these bodies unite and travel together, what is the common velocity after the shock?

$$\text{Mass of A} = \frac{772}{32\frac{1}{6}} = 24 \qquad (m);$$

$$\text{Mass of B} = \frac{965}{32\frac{1}{6}} = 30 \qquad (\text{M}).$$

Then according to (I.) the common velocity

$$u = \frac{24 \times 7 + 30 \times 4}{24 + 30} = 5\tfrac{1}{3} \text{ ft.}$$

Now had we employed the weights instead of the masses the result would have been the same, and much unnecessary work avoided;

$$u = \frac{772 \times 7 + 965 \times 4}{772 + 965} = 5\tfrac{1}{3} \text{ ft.}$$

In this way we avoid employing the quantity g. The same result would be obtained if the lb. weight happened to be too light or too heavy. Suppose in the present case the standard pound to be $\frac{1}{193}$ part too light, so that A's correct weight would be $772 + \frac{772}{193} = 776$ lbs., and B's $965 \times \frac{965}{193} = 970$ lbs. Putting these weights for the masses in (I.),

$$u = \frac{776 \times 7 + 970 \times 4}{776 + 970} = 5\tfrac{1}{3} \text{ feet.}$$

If the body shocked was at rest, we should have $\text{V} = o$, and (I.) is reduced to

$$u = \frac{m v}{m + \text{M}}. \qquad \text{(II.)}$$

or, substituting the weights, w, W, for the masses (II.) becomes

$$u = \frac{w\,v}{w + \mathrm{W}}.$$

If two bodies strike against each other in opposite directions, a similarity of action exists; but then, at the end of the compression, either the bodies are both brought to a state of rest, and we have

$$m\,v = \mathrm{M\,V}, \text{ and } u = o, \text{ in (I.)},$$

or one of the two goes backward, and they proceed with a common velocity, u.

If it is the body M which goes backwards, the quantity of motion lost by the body m is $m\ (v - u)$, and the quantity of motion developed during the period of compression by the forces of reaction upon the body M is composed of that which has been destroyed, or M V, plus that imparted in an opposite direction, M u, and because the quantities of motion developed on both sides upon each of the bodies should be equal; therefore,

$$m\ (v - u) = \mathrm{M}\ (\mathrm{V} + u);$$

$$\therefore \quad u = \frac{m\,v - \mathrm{M\,V}}{m + \mathrm{M}} \qquad \text{(III.)}$$

If the weights of the bodies, w, W, be substituted for the masses in (III.), the value of u will not be altered, and becomes

$$u = \frac{w\,v - \mathrm{W\,V}.}{w + \mathrm{W}} \qquad \text{(IV.)}$$

Divide the numerator and denominator of (IV.) by w and it becomes,

$$u = \frac{v - \frac{\mathrm{W}}{w}\mathrm{V}}{1 + \frac{\mathrm{W}}{w}}; \qquad \text{(V.)}$$

a formula in which we see that the velocity of the impinging body will be so much the less changed as its weight, w, is greater in its ratio with that of the body shocked. Hence in machines working by shocks we must increase the weight of the impinging pieces in their ratio to the pieces shocked, in a ratio so much greater as it is desired to maintain a greater regularity of motion. If the body shocked is at rest, such as a pile driven by a ram, we have $V=o$; the common velocity of the descent of the pile and the ram after the shock is

$$u=\frac{1}{1+\frac{W}{w}}v,$$

which shows that this velocity will differ so much the less from that of the arrival of the ram upon the head of the pile as the weight w of the ram is greater in its ratio with that of the pile. It is best in this case, then, to increase the weight of the ram rather than its velocity; for the work employed to raise it increases only with its weight, while its work will be increased proportionally to the height of elevation, or to the square velocity of its descent.

Ques. 41. Let a body, A, weighing 1351 lbs. (w) move with a velocity of 10 feet a second and impinge on a body, B, weighing 579 lbs. (W) moving with a velocity of 14 feet a second in an opposite direction in a straight line joining the centres of gravity of A and B; after compression, suppose these bodies to unite and travel together, what is the common velocity after the shock?

$$\text{Mass of A}=\frac{1351}{32\frac{1}{6}}=42; \qquad (m),$$

$$\text{Mass of B}=\frac{579}{32\frac{1}{6}}=18 \qquad (M).$$

From (III.) we find the common velocity u,

$$u=\frac{42\times 10-18\times 14}{18+42}=2\cdot 8 \text{ feet.}$$

Had we employed the weights of the bodies instead of the masses of A and B, the result, as in the former case, would not be altered. Thus,

$$u=\frac{1351\times 10-579\times 14}{1351+579}=2\cdot 8 \text{ feet a second.}$$

In this way we avoid employing the quantity $g=32\frac{1}{6}$, or any other value. The same result would be obtained if the pound weight that measured the weight of the bodies happened to be too heavy or too light. Suppose, in the present case, the standard pound to be $\frac{1}{193}$ part too heavy; then the correct weight of A would be $1351 - \frac{1351}{193} =$ 1344 lbs., and the correct weight of B would be $579 - \frac{579}{193} = 576$ lbs.

$$\therefore \quad u = \frac{1344 \times 10 - 576 \times 14}{1344 + 576} = 2{\cdot}8 \text{ feet.}$$

However, in measuring units of work, the pound weight must be correct.

The preceding considerations were proved by direct experiments by the justly celebrated French engineer, General Arthur Morin. He employed the following apparatus: A wooden box, in which was placed successively clay, more or less soft, sand, pieces of wood, &c., the whole suspended to a dynamometer having a style and turning plate. The plate was impressed with a uniform motion, which was transmitted by a weight and regulated by a fan fly-wheel. When the box was immovable, the resistance of the dynamometer was in equilibrium with the weight of the box and its contents, and the curve of flexure traced by the style upon the plate was a circle. The impinging body was a cannon-ball held by thongs, opening at pleasure, and when it struck the materials in the box it caused compression, immediately after which the two bodies fell together with a common velocity. It is easy to compare in each case the results of these experiments with those of theory. Some of these comparisons are presented here:

Experiments upon the Transmission of Motion by the Shock of a Spherical Projectile falling upon a Box filled with Clay, or containing Pieces of Wood.

Weight of the box and its load.	Weight of sphere.	Total weight.	Weight of fall of sphere.	Velocity due to height of fall.	Velocity imported to box,		Approximate duration of transmission of motion.	
					by theory.	by experiment.		
W.	w.	$W + w$.	h.					
lbs.	lbs.	lbs.	ft.	ft.	ft.	ft.	sec.	
132·84	12·23	146·08	1·31	9·19	0·88	0·85	·012	Experiments made with clay, whose resistance to penetration of projectiles with small velocity was 6080 lbs. the square inch.
134·74		147·97	1·64	10·28	0·92	0·93	·020	
			0·65	6·50	1·08	1·08	·019	
132·84	26·64	159·28	0·98	7·96	1·40	1·31	·021	
			1·31	9·19	1·52	1·51	·024	
132·84	44·73	177·57	0·65	6·50	1·64	1·61	·020	
147·82	12·23	161·05	0·65	6·50	0·53	0·54	·063	Experiments made with clay, whose resistance to penetration was 346 lbs. to a square foot.
147·82	44·73	192·54	0·65	6·50	1·51	1·44	·072	
48·45	26·64	74·85	0·65	6·50	2·27	2·16	·0075	
48·45	26·64	76·85	0·98	7·96	2·79	2·75	·0074	
48·45	44·73	93·14	0·33	4·59	2·20	2·26	·0060	

Common Geometrical Problem, in which none of the principles of Mechanics are involved.

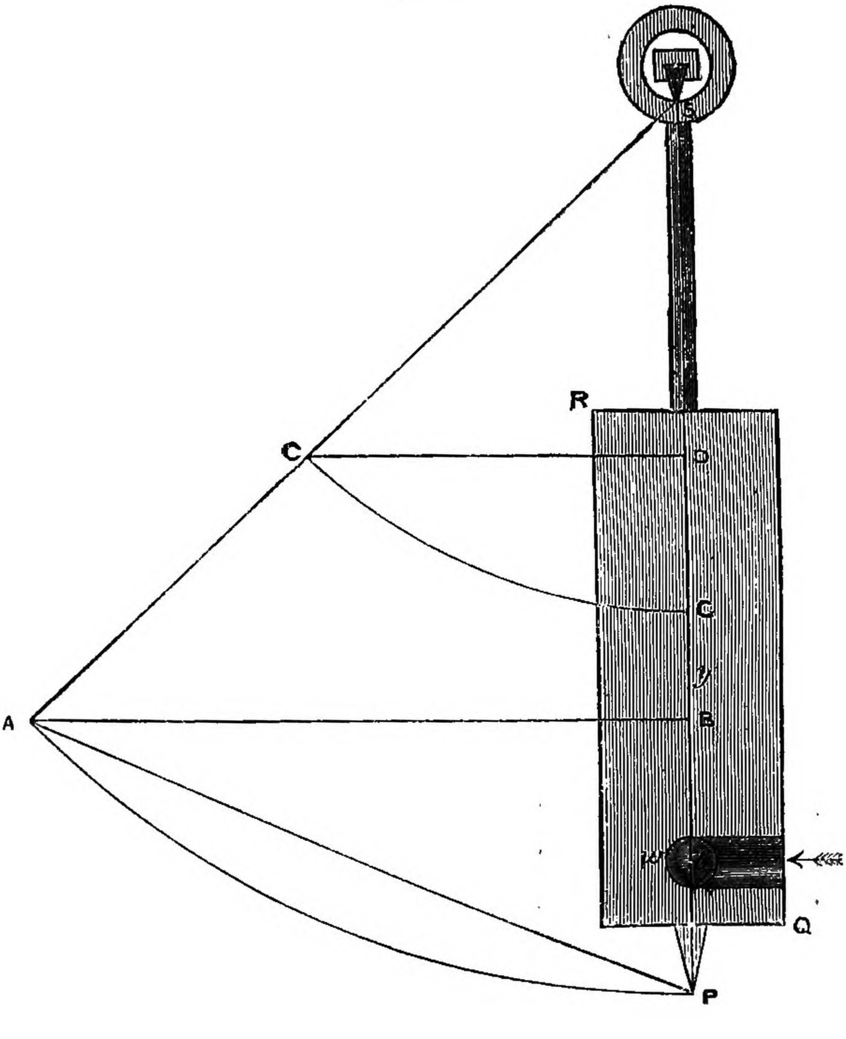
Given S P = 13·52 ft. the radius of the index P of a ballistic pendulum which shows the range of the recoil by marking soft compressible matter placed in a box. A P the chord of the arc of recoil = 10·4 ft., S G = 10·14 ft., the distance of the centre of gravity, G, of the pendulum, P Q R S, and ball *w*. It is required to find the height, D G, that the centre of gravity is raised when P moves from P to A?

Draw A B and C D perpendicular to S P; then A P^2 divided by twice S P = B P, Fig. 60.

$$\therefore \quad BP = \frac{10{\cdot}4^2}{27{\cdot}04} = 4 \text{ ft.}$$

And S P : P B :: S G : D G, that is
13·52 : 4 :: 10·14 : 3 ft. = D G.

Fig. 60.



Mechanical Problem.

Let y be the centre of gyration of the pendulum P Q R S, Fig. 60, without the ball, $Sy=10$ ft.; if $W=$ the weight of the pendulum, and $w=$ the weight of the ball, b, the point where the ball impinges, $bS=12$ ft., what weight must be placed at b, so that the pendulum would receive the same motion as when its weight is placed at y, its centre of gyration; the position of the centre of gyration of the pendulum and ball together is also required?

We have before shown that if a weight $=\dfrac{W\times Sy^2}{Sb^2}$ be placed at b, the pendulum would receive the same motion from a blow as when the weight, W, is placed at y. Now if the weight W of the pendulum = 9 tons = 20160 lbs.,

$$\frac{W\times Sy^2}{Sb^2}=\frac{20160\times 10^2}{12^2}=14000 \text{ lbs.}$$

Suppose w the weight of the ball $=20$ lbs., then if $z=$ the distance of the centre of gyration of both the pendulum and ball together from the point of suspension S, from what has been shown, page 104,

$$z^2=\frac{W\times Sy^2+w\times Sb^2}{W+w}=$$

$$\frac{20160\times 10^2+20\times 12^2}{20160+20}=100{\cdot}0436.$$

$\therefore \quad z=10{\cdot}0022$, so that the position of the centre of gyration is not much changed by the weight of the ball.

Ques. 42. A ballistic pendulum, P Q R S, Fig. 60, is struck by a cast-iron round-shot b, weighing w lbs. $=20$ lbs. with a velocity v; the pendulum being at rest weighing W lbs. $=$ 20160 lbs.; S y, the distance of the centre of gyration of the pendulum from the point of suspension $=10$ ft.; S b, the distance from the point of suspension to where the ball lodges $=12$ ft. Required the velocity of the point where the ball strikes and of the centre of gyration of both pendulum and ball combined?

By the last problem, if 14,000 lbs. be placed at b, the pendulum would receive the same motion as when the weight of the pendulum $W = 20160$ lbs. is placed at y, its centre of gyration. We may, therefore, consider the body w to impinge directly with a velocity v upon the body weighing 14000 lbs. at rest, and the two bodies will move on together after impact in the same manner as if they were inelastic. Let x be the common velocity after the stroke, then, since the *momentum* or quantity of motion is the same before and after impact, we have

$$x(w+14000)=wv;$$

$$\therefore \quad x=\frac{wv}{w+14000}=\frac{20v}{14020}.$$

By the last problem the distance of the centre of gyration of ball and pendulum from the point of suspension $=$ 10·0022 ft.

$$\therefore \quad 12 : 10\cdot0022 :: \frac{20v}{14020} : \frac{200\cdot044v}{168240},$$

the velocity of the centre of gyration of the bodies combined.

Ques. 43. A ball weighing 20 lbs. strikes a ballistic pendulum, P Q R S, Fig. 60, weighing 9 tons, at a distance of 12 ft. from the point of suspension, making the length of the chord of recoil $=$ A P $=$ 10·4 ft.; S P, the radius of the index P $=$ 13·52 ft. The distance S G $=$ 8·45 ft., G being the centre of gravity of the ball and pendulum combined. S y $=$ 10 ft., y being the centre of gyration of the pendulum without the ball. Required v, the velocity with which the ball strikes the pendulum?

By the preceding geometrical problem, P B $=$ 4 ft., then 13·52 : 4 : : 8·45 : 2·5 ft. $=$ G D.

From the last question $\frac{200\cdot044v}{168240}$ $=$ the velocity of the centre of gyration of the ball and pendulum combined. Therefore, the units of work developed by the ball $=$

$$\left(\frac{200\cdot044v}{168240}\right)^2 \times \frac{w+W}{64\frac{1}{3}}.$$

These units of work are wholly expended in raising the centre of gravity from G to D$=2\frac{1}{2}$ feet. But the units of work required to raise the centre of gravity from G to D is also$=$

$$2\tfrac{1}{2}\times(w+\mathrm{W});$$

$$\therefore\quad\left(\frac{200{\cdot}044\,v}{168240}\right)^2\times\frac{w+\mathrm{W}}{64\frac{1}{3}}=2\tfrac{1}{2}\,(w+\mathrm{W});$$

$$\therefore\quad\left(\frac{200{\cdot}044\,v}{168240}\right)^2=2\tfrac{1}{2}\times 64\tfrac{1}{3};$$

$$\therefore\quad v^2=\frac{5\times 192}{6}\left(\frac{168240}{200{\cdot}044}\right)^2$$

$$\therefore\quad v=\sqrt{\frac{965}{6}}\times\frac{168240}{200{\cdot}044}=10665{\cdot}75\text{ feet.}$$

This problem is distorted to show how the principle of work will apply without considering the oscillations of the pendulum or involving the centre of oscillation.

SHOCK OF TWO ELASTIC BODIES.

If we suppose that the two bodies under consideration are perfectly elastic, moving in the same direction, the effects of compression will be at first the same as in one of the preceding cases, pp. 134 to 140, and at the end of this time the body m will have lost a velocity, $v-u$, and a quantity of motion $m\cdot(v-u)$, and the body M will have gained a quantity of motion $\mathrm{M}\,(u-\mathrm{V})$, and these quantities being equal, we have, as before explained, the common velocity u, at the end of the compression,

$$u=\frac{m\,v+\mathrm{M}\,\mathrm{V}}{m+\mathrm{M}}.$$

But, after the instant of greatest compression, the elastic bodies regain their primitive form, and in the return to it develop, if the elasticity is powerful, efforts equal to their resistance or compression, and consequently destroy or impart quantities of motion equal to those which they have previously destroyed or imparted. It follows from this that in the unbending of the molecular springs the body m will further lose a velocity$=v-u$, and that its final velocity will be

$$v-2\,(v-u)=2\,u-v,$$

and the body M will receive a new increase of velocity equal to $u-V$, and will then have a final velocity equal to

$$V+2\,(u-V)=2\,u-V.$$

If the body M were at rest at the beginning, supposing it perfectly elastic, it will then receive a velocity$=2\,u$, as $V=o$. That is to say, twice that imparted to a soft body under the same circumstances, or

$$\frac{2\,m\,v}{m+M}.$$

These reasonings relative to soft and elastic bodies presuppose the existence of bodies deprived of all elasticity, and others endowed with perfect elasticity. Neither of these hypotheses is exact; and, according to circumstances in which they are placed, a body may act as if deprived of all elasticity, or as if possessed of only a partial elasticity. So also a body which, in certain conditions, acts as if it were perfectly elastic, will only appear as if partially so in other cases. We will here cite some examples, the results of experiments analogous to the preceding theory, made by Morin, and which were effected by placing at the bottom of a movable box a plate of cast iron, upon which fell a spherical body.

MORIN'S EXPERIMENTS UPON THE TRANSMISSION OF MOTION BY THE SHOCK OF A SPHERICAL PROJECTILE FALLING UPON A CAST-IRON PLATE.

Weight of the box and its load.	Weight of the iron ball.	Total weight.	Height of the fall of the ball.	Velocity due to the height.	Velocity imparted to the box, by Theory.	Velocity imparted to the box, by Experience.	Approximate duration of the transmission.
w	W	$w+W$	h		$2\,u$		
135	13·23	148·23	1·31	9·19	1·64	1·64	·0085
135	13·23	148·23	1·64	10·28	1·84	1·87	·0081
135	13·23	148·23	1·97	11·26	2·07	2·05	·0080
135	26·44	161·44	1·31	9·19	3·01	2·98	·0065
135	26·44	161·44	1·64	10·28	3·36	3·44	·0075

The results recorded in this table show that the cast-iron plate shocked acted as a body perfectly elastic. But it is proper here to make some important remarks. The projectile, which, had it been in the condition of a perfectly elastic body as well as the parts of the plate with which it came in immediate contact, would have risen a height corresponding to the velocity $2\,u-v$, did not by any means attain this height. This proves that the intensity of the shock in these experiments had changed, in a great measure, the elasticity of the molecular springs of the parts in contact, while the elasticity of flexure, or of the general form of the plate, had not been altered. We see by this, that although bodies endowed with a certain elasticity apparently resume their primitive form, there is, nearly in every case, a notable loss of work produced by the shock, by reason of the more or less complete alteration of its elasticity.

QUANTITY OF MOTION IMPARTED BY A CONSTANT FORCE.

When the force is constant, let us call it F, and the time it is in operation, long or short, let it be called t. Whence

$$\mathrm{F}\,t=m\,v;$$

$$\therefore\quad \mathrm{F}=\frac{m}{t}\,v.$$

This expression shows that the effort required to impart or destroy a quantity of motion $m\,v$ is so much the greater as the time employed is less; and since the reciprocal action of bodies is more rapid, compared to the spaces described, their compressions, flexures, and penetrations are less for the same quantity of motion destroyed. We have here explained why it is that the shock of hard bodies, the transmission or destruction of motion by bodies slightly flexible, compressible, or extensible, occasion such great efforts and such ruptures and accidents; and how it is, on the other hand, by the interposition of soft and compressible bodies that the intensity of efforts and their consequences are so much diminished. We see by the expression $F=\frac{m}{t}v$ that a finite velocity v could never impart in an infinitely small time to a mass m, except by an infinite effort, which shows the error in the hypothesis of the instantaneous transmission of motion by forces, to which we are then compelled to give a special name, and thus suppose a special nature in calling them *forces of percussion.* This error is often too explicitly admitted in the teachings of rational mechanics. Nothing like an instantaneous operation really occurs in nature; quantities of motion are imparted and destroyed in greater or less periods of time, sometimes, indeed, imperceptible to our senses and means of observation, but never instantaneous.

Familiar examples will enable the student better to appreciate this matter. We have before alluded to the quantity of motion imparted to a ball upon which the force of gunpowder imparted a given velocity. We shall here take another example of this kind.

Ques. 44. Let it be required to find the force or effort of the gunpowder that impresses a velocity of 1500 feet a second on a ball weighing 96·5 lbs.; supposing the velocity to be communicated in $\frac{1}{75}$ part of a second, or while the ball passes through the barrel whose length is 20 feet?

$$m=\frac{96{\cdot}5}{32\frac{1}{6}}=\frac{W}{g}=3$$

$$\text{But } F=\frac{m}{t}v=3\times 75\times 1500=337500 \text{ lbs.},$$

for to divide by the time $t=\frac{1}{75}$ we multiply by 75. This gives an idea of the enormous efforts developed by the gunpowder.

Ques. 45. Suppose two horses to impress upon a coach, weighing 3 tons 17 cwts. 14 lbs., a velocity of 6 miles an hour, what force must these horses develop to start this load in two seconds, with a velocity of 8·8 feet a second, or 6 miles an hour?

$$3 \text{ tons } 17 \text{ cwt. } 14 \text{ lbs.} = 8685 \text{ lbs.}$$

$$m = \frac{W}{g} = \frac{8685}{32\frac{1}{6}} = 270.$$

$$F = \frac{m}{t} \times v = \frac{270}{2} \times 8{\cdot}8 = 1188 \text{ lbs.},$$

neglecting the friction of the ground and the friction of the wheel-boxes, which in ordinary cases would require an effort of

$$\frac{8685}{30} = 289{\cdot}5 \text{ lbs., or } 144{\cdot}75 \text{ to each horse, and}$$

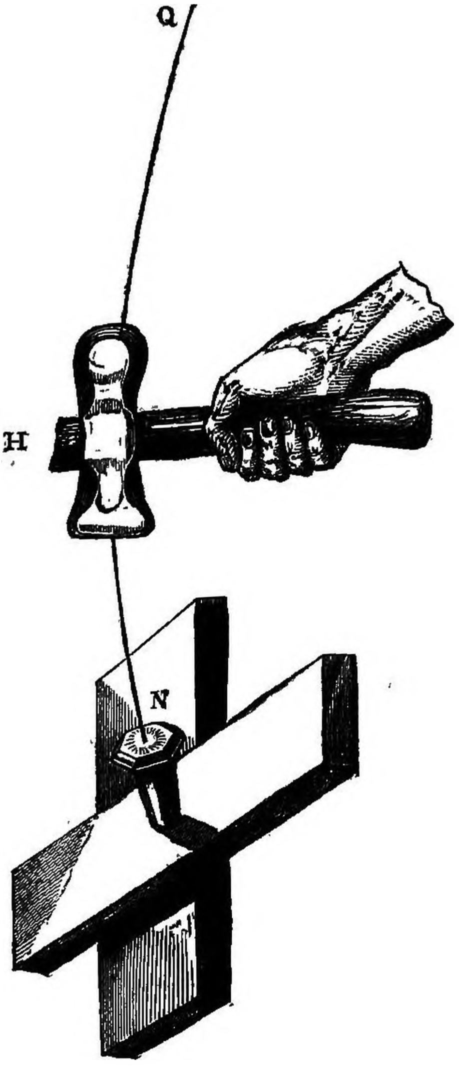
$$\frac{1188}{2} + 144{\cdot}75 = 738{\cdot}75 \text{ lbs.}$$

We see, in this case, that to impart this velocity in two seconds, each horse must develop a mean effort of 738·75 lbs., which is more than 18 times the mean effort to be exerted after the velocity has been once acquired; for a horse can only use a traction of $41\frac{2}{3}$, going at the rate of 5 miles an hour. (See page 12.)

It may be remarked here, that the breaking of traces, of swing bars, wounds upon the breasts, and straining of hams, arise from the great rapidity of the destruction of the quantity of motion impressed by the horses upon their own masses by the resistance and reaction of the inertia of the vehicle; whence the necessity of starting with slack traces, and of warning and urging the horses gently with the voice. Similar effects are produced in starting and stopping railroad trains; and in seeking the means of promptly checking these enormous masses, we must bear in mind that too sudden changes of velocity are dangerous to the passengers. Finally, the means adopted for the connexion of machines, or for rapid transmission of motion to them, should be disposed or proportioned agreeably to these ideas.

Ques. 46. A hammer, H, Fig. 61, strikes a nail, N, and drives it $\frac{1}{4}$ of an inch, the hammer weighs 11·58 lbs., and in delivering the blow it passes over the space Q N = 10 feet in a second; required the force in pounds delivered by the hammer upon the head of the nail?

Fig. 61.



$$10\text{ ft.} : 1'' :: \tfrac{1}{48}\text{ ft.} : \tfrac{1}{480}''$$

Hence the time occupied in driving the nail $\frac{1}{4}$ of an inch cannot be less than $\frac{1}{480}$ of a second $= t$.

$$m = \frac{11\cdot58}{32\frac{1}{6}} = \cdot36;$$

$$\because \quad F = \frac{m}{t} \times v = \cdot36 \times 480 \times 10 = 1728 \text{ lbs.}$$

$\therefore$ The force in pounds delivered upon the head of the nail is nearly = a ton.

The student should be careful in making use of what is here termed the quantity of motion, or *momentum*, for when we know the product of the mass of a body and the velocity imparted or taken from it, we have the measure of effect produced by the force during the period of action; but we see that this measure cannot be taken as a term of comparison except for analogous cases, where the velocities are really imparted or destroyed by the force, and it does not follow that the product $F t$ of the force, by its period of action (equal, when there is a change in the state of motion, to the quantity of motion imparted or destroyed), should always serve as a measure of the effect of forces, as is sometimes admitted for certain instruments and certain kinds of work. Indeed, it is readily seen that an effort may continue a long time without producing a mechanical effect. Thus, horses pulling upon a mired waggon, without starting it, develop considerable efforts, which, multiplied by the period of their action, would give an enormous product without any useful effort resulting in any mechanical work, and nothing but the fatigue and exhaustion of the horses. Take, for example, the draught of a plough, which in strong earth requires a mean total force of 794 lbs. Suppose the furrow to be 400 feet long, the horses in one take 100″ and in the other 200″ to plough it. We shall have for the first case,

$F t = 794 \times 100'' = 79400$; and in the second,

$F t = 794 \text{ ft.} \times 200'' = 158800$; and yet in both cases they have accomplished the same work.

An instrument giving the product of efforts by the times or periods of duration would by no means lead to an exact appreciation of the mechanical effects produced. The true measure of these effects is, the method established in this work, the product of the effort exerted by the path described in its duration, which we estimate in **UNITS OF WORK.**

Important Observation.

It should be further observed, that it is only in the case of a constant effort acting during a time $t = 1''$ that we can take the product $m v$ for the measure of the effort F, and then we have

$$F = M v = \frac{W}{g} v, \text{ or the proportion}$$

$$F : W :: v : g.$$

But in cases of variable efforts, the same mode of measurement does not apply for finite times, for forces varying according to very different laws may in the same time impart equal quantities of motion to the same body or to different bodies. The formula $F = M v$ will only give then the value of a mean constant effort capable of imparting in the same time the same quantity of motion.

VELOCITY ACQUIRED BY A BODY DESCENDING AN INCLINED PLANE.

If a body freely descends an inclined plane by the force of gravity alone, the units of work accumulated in the body will be that due to the perpendicular elevation, without regard to the angle or curve of the path. Consequently, the velocity of the body will be that which it should acquire in falling freely through the perpendicular height.

Ques. 47. What velocity will a body acquire in descending an inclined plane whose perpendicular height is = 579 feet?

Putting t for the time of descent,

$$t^2\times16\tfrac{1}{12}=579,$$

$$\therefore\quad t^2=36, \text{ and}$$

$$t=6 \text{ seconds};$$

$\therefore$ the velocity acquired at the end will be $32\frac{1}{6}\times6=193$ feet.

Ques. 48. A train of 120 tons descends an incline of 1000 feet, having a total rise of 20 feet; what will be the velocity acquired by the train, supposing the friction to be 7 lbs. a ton?

$$120 \text{ tons}=268800 \text{ lbs.}$$

The units of work due to gravity will be

$$268800\times20=5376000.$$

Units of work destroyed by the friction $=$

$$120\times7\times1000=840000.$$

Therefore the units of work in the train when it arrives at the bottom of the incline will be

$$2688000-840000=1848000.$$

Putting v for the velocity acquired, we have

$$\frac{v^2\times268800}{64\frac{1}{3}}=1848000.$$

$$\therefore\quad v^2=\frac{1188880}{2688}=442\cdot291666.$$

$$\therefore\quad v=21 \text{ feet, nearly.}$$

THE RESISTANCE OF FLUIDS.

The resistance of fluids to an immersed body is occasioned by the force required to displace the fluid. It is evident that the fluid displaced must have the same velocity given to it as that of the moving body; it follows, therefore, that *the work* destroyed by the fluid must be equal to the *accumulated work* in the fluid. Take, for example, the front of a body$=$201 square feet presented perpendicularly to the line of motion, the weight of a cubic foot of the fluid$=62\frac{1}{2}$ lbs., and the velocity of the body 10 miles an hour$=14\frac{2}{3}$ feet a second; then the weight of the fluid displaced every second=

$$201\times14\tfrac{2}{3}\times62\tfrac{1}{2}=182776 \text{ lbs.}$$

But this weight has a velocity of $14\frac{2}{3}$ a second;

$\therefore$ the *units of work* expended in the displacement will be

$$\frac{(14\frac{2}{3})^2\times182776}{64\frac{1}{3}},$$

since these units of work have been destroyed, while the body moved over $14\frac{2}{3}$ feet; if x be put for the resistance in pounds, this work will also be represented by

$$x\times14\tfrac{2}{3}.$$

$$\therefore \quad x\times14\tfrac{2}{3}=\frac{(14\frac{2}{3})^2\times182776}{64\frac{1}{3}};$$

$$\therefore \quad x=\frac{14\frac{2}{3}\times182776}{64\frac{1}{3}}=41669 \text{ lbs.}$$

The value of x may be put under the form

$$x=(14\tfrac{2}{3})^2\times201\times\frac{62\frac{1}{2}}{64\frac{1}{3}}.$$

It therefore appears that the resistance increases with the square of the velocity, as well as with the extent of the surface presented to the fluid. In extreme velocities this law does not hold strictly true, as the friction of the fluid along the sides of the body has considerable effect. It appears also, from certain railroad experiments, that the resistance of the atmosphere to the motion of the train depends chiefly upon the length of the train, and not so much upon the extent of the frontage of the carriages.

THE EFFLUX OF FLUIDS.

When an aperture is made in the side of a vessel, constantly full of water, then the *accumulated work* of the fluid, on being discharged through the aperture, is equal to *the work* which the force of gravity would perform upon it while descending a space equal to the perpendicular depth of the aperture. Consequently, the velocity of the water as it leaves the orifice is equal to the velocity which a body would acquire in falling freely through a space equal to the perpendicular depth of the aperture.

Ques. 49. The cross sectional area of pipe is 36 square inches; how many cubic feet of water will it discharge a minute from a vessel kept constantly full, the depth of the discharge-pipe being 21 feet?

Let v be the velocity of the discharge, and w the weight in lbs. of the water discharged; then the *accumulated work* in w as it is discharged will be

$$\frac{v^2 \times w}{64\frac{1}{3}}.$$

But the *work* in descending 21 feet is equal to

$$21 \times w;$$

$$\therefore \quad \frac{v^2 \times w}{64\frac{1}{3}} = 21 \times w;$$

$$\therefore \quad v^2 = 21 \times 64\tfrac{1}{3} = 1351;$$

$$\therefore \quad v = \sqrt{1351} = 36{\cdot}756 \text{ feet.}$$

Then the discharge in cubic feet a minute will be

$$\frac{36}{144} \times 60 \times 36{\cdot}756 = 551\tfrac{1}{3} \text{ cubic feet, nearly.}$$

Ques. 50. How many cubic feet of water will a pipe 2 square inches of cross section discharge in a minute, the vessel being kept constantly full, and 5000 lbs. placed on a piston fitting the cistern in contact with the water; the cross sectional area of the piston 4 square feet, the depth of the discharge-pipe 10 feet?

The effect of the pressure on the piston is equivalent to that of a column of water that would produce the given pressure,

$$\therefore \quad \frac{5000}{4 \times 62{\cdot}5} = 20 \text{ feet additional height.}$$

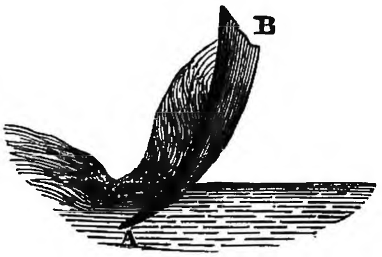
The height of the column of water producing the same pressure $= 20 + 10 = 30$.

$\therefore$ Velocity $= \sqrt{30 \times 64\frac{1}{3}} = 43{\cdot}93$ feet a second, and the discharge in a minute will be $\frac{2}{144} \times 43{\cdot}93 \times 60 = 36{\cdot}6$ cubic feet.

OF PONCELET'S WATER-WHEEL.

In the common under-shot water-wheel the paddles are flat, whereas in Poncelet's wheel they have a curved shape, A B; so that the direction of the curve at A, where the water meets the paddle, is the same as the direction of the stream.

Fig. 62.



By this ingenious contrivance the water rolls up the curved incline A B without meeting with any sudden obstruction calculated to occasion a loss of work. The channel has a depression at the point where the water falls from the paddles. Let V be the velocity of the stream, and v that of the wheel; then, since the point A of the paddle is moving away from the stream, the water will flow upon the paddle with the velocity $V-v$, and will continue to run up the curved line until it has lost its motion; it will then descend, acquiring in its descent the same velocity as that which it had in its ascent, but in a contrary direction. If the wheel were not to move during its return, $V-v$ would be exactly the velocity of discharge from right to left; but the paddle is moving the water from left to right with the velocity v, therefore the absolute velocity of the water upon leaving the paddle will be $V-v-v=V-2v$.

Now all the work will have been taken out of the water when its motion upon leaving the paddle is nothing—that is, when $V-2v=0$,

$$\text{or } V=2v.$$

In this case the water, having lost all its motion, will simply drop from the paddle, and the work done upon the wheel will be equal to the work accumulated in the water of the stream. Moreover, it appears that the maximum condition is fulfilled when the velocity of the stream is double that of the wheel. However, the distinguished inventor states that, in practice, the velocity of the water, in order to procure its maximum effect, ought to be $2\frac{1}{2}$ times that of the wheel, and that the *modulus* of the wheel is about $\frac{7}{10}$. This wheel, other things being the same, will perform about twice the work of the common under-shot wheel.

When there is motion in the water after leaving the wheel, the work accumulated in the water must be calculated, and subtracted from the whole work originally in it, in order to obtain the work done upon the wheel.

$$\frac{W \times V^2}{2g} = U + \frac{W (V - 2v)^2}{2g};$$

$$\therefore \quad U = \frac{2W}{g} (V - v) v.$$

Ques. 51. Suppose 193 cubic feet of water flow upon the paddles of a Poncelet wheel a second, with the velocity of 8 feet, when the wheel moves at the rate of 4 feet a second; required the horse power of the wheel?

This is obviously a case of maximum work. Weight of the water flowing a second

$$= 193 \times 62{\cdot}5 = 12062{\cdot}5 \text{ lbs.}$$

Work each second

$$= \frac{8^2 \times 12062{\cdot}5}{64\frac{1}{3}} = 12000;$$

Horse power

$$= \frac{12000 \times 60}{33000} = 21^h{\cdot}818.$$

No account is taken of friction and other resistances.

Ques. 52. Required the same as in the preceding example, when the quantity of water is 386 cubic feet a second, the velocity of the stream 8 feet, and of the wheel 2 feet?

The velocity with which the water leaves this wheel will be $8 - (2 \times 2) = 4$ feet.

Weight of the water acting upon the wheel each second $= 386 \times 62{\cdot}5$ lbs. $= W$.

$$(V - v)\, v = (8 - 2) \times 2 = 12.$$

Hence the units of work developed by the wheel in a second will be

$$\frac{2\,\mathbf{W}}{g}\times(\mathbf{V}-v)\,v=\frac{2\,(386\times62{\cdot}5)}{32\frac{1}{6}}\times12=18000'';$$

$\therefore$ the horse power will be

$$\frac{18000''\times60}{33000'}=32\tfrac{8}{11}{}^{\text{h}}.$$

FIRE-ENGINE.

In this machine water is forced through a tube with a velocity necessary to reach a certain height. When the quantity of water that passes through the nozzle in a second is given, and the area of the cross section, the velocity is readily calculated, and also the power of the machine.

Ques. 53. How many men are required to work a fire-engine, allowing each man to perform 2500 units of work in a minute; the area of the cross section of the nozzle = $\frac{3}{4}$ of a square inch, when 100 gallons, each containing 277·274 cubic inches, have to be discharged in a minute?

$\frac{3}{4}$ of a square inch $=\frac{1}{192}$ square foot.

Cubic feet of water discharged in a second $=\frac{27727{\cdot}4}{60\times1728}$; hence the velocity in feet a second will be $\frac{2772{\cdot}74}{6\times1728}\times\frac{192}{1}=51{\cdot}34$ feet a second.

100 imperial gallons = 1000 lbs. of water a minute;

$\therefore$ the accumulated work in a second

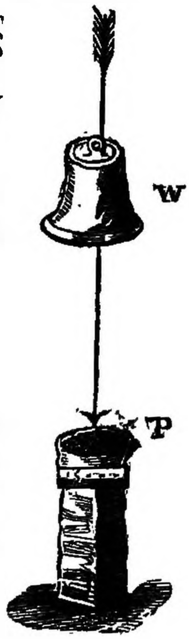
$$=\frac{(51{\cdot}34)^2}{64\frac{1}{3}}\times\frac{1000''}{60}.$$

The accumulated work in a minute will be

$$\frac{(51{\cdot}34)^2}{64\frac{1}{3}}\times1000=40970{\cdot}9'.$$

$\frac{40970{\cdot}9}{2500}=16{\cdot}388$; so that 17 men will be required to work this fire-engine.

Fig. 63.



Ques. 54. Let a ram, W, Fig. 63, weighing 1158 lbs. fall 24 feet, and drive the pile P 4 inches. What force in lbs. is developed by the ram on the head of the pile?

Units of work expended in raising the ram= **24×1158**; putting v for the velocity acquired by the ram in falling 24 feet, then

$$\frac{v^2 \times 1158}{64\frac{1}{3}} = 24 \times 1158.$$

$\therefore \quad v=\sqrt{64\frac{1}{3} \times 24}=39{\cdot}2938$ feet a second. The time that the pile is penetrating 4 inches cannot be less than $\frac{1}{117{\cdot}8814}$ part of a second; $F\,t = m\,v$, and $F=\frac{m}{t}v$.

$$m=\frac{1158}{32\frac{1}{6}}=36.$$

$$\therefore \quad F=36 \times 117{\cdot}8814 \times 39{\cdot}2938=166852 \text{ lbs.}$$

Definition.—The centre of oscillation, O, Fig. 64, is that point of a body, Q R, in which, if the whole mass was collected, it would vibrate through a given angle about a horizontal axis, P T, by the force of gravity as before the matter was collected.

If G be the centre of gravity, y the centre of gyration, then the position of the centre of oscillation is instantly found; for, if P be the centre of motion,

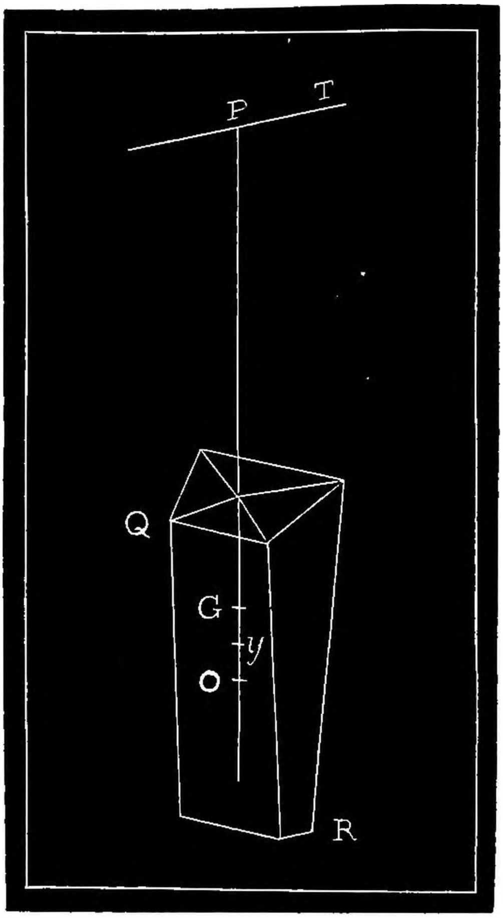
$$PO = \frac{(Py)^2}{PG}.$$

If $PG=18$, and $Py=24$, then

$$PO=\frac{576}{18}=32.$$

The *centre of percussion* is the same as the centre of oscillation.

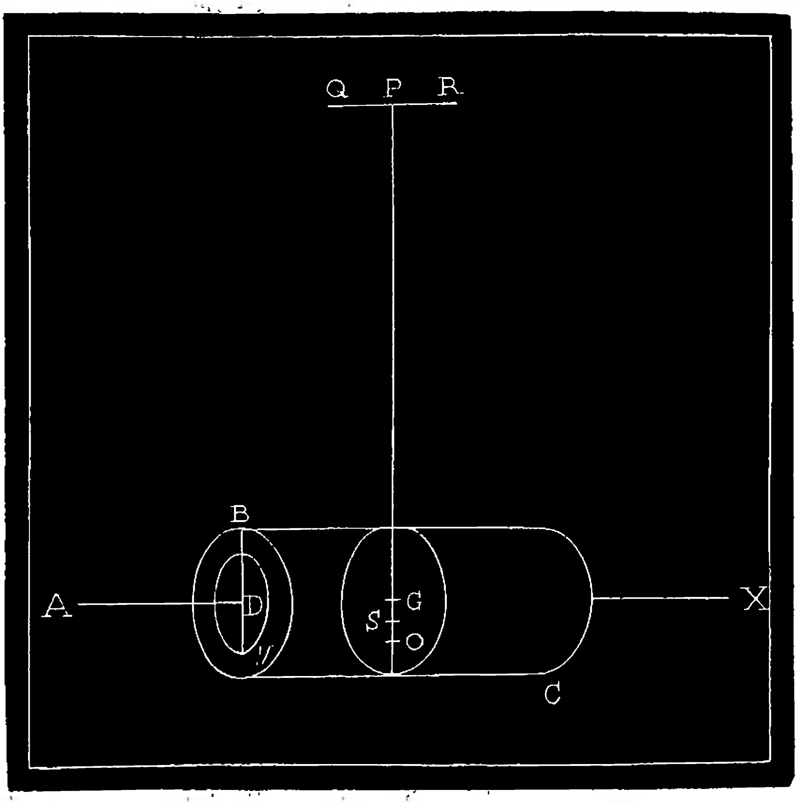
Fig. 64.



If y^2 be the square of the radius of gyration around an axis passing in any direction through the centre of gravity of a body, and x the perpendicular distance between this axis and another parallel to it, then $y^2 + x^2 =$ the square of the radius of gyration through the second axis.

Ques. 55. A cylinder, B C, Fig. 65, moves round a fixed horizontal axis, Q R, parallel to the axis of the cylinder, A X; required the position of the centre of gyration, S, and the centre of oscillation, O, with respect to the axis Q R; the distance from the point of suspension, P, to the centre of gravity G = 45 inches (a), the radius of the cylinder, D B = 6 inches (r)?

Fig. 65.



When the cylinder revolves round the axis A X, the square of the radius of gyration $D\,y = \frac{1}{2}r^2$; but the axis Q R is parallel to A X; therefore, if S be the centre of gyration for this axis,

$$S\,P = \sqrt{a^2 + \tfrac{1}{2}r^2} = \sqrt{45^2 + \tfrac{36}{2}} = 45{\cdot}2, \text{ nearly.}$$

The distance from P to O, the centre of oscillation,

$$= \frac{(P\,S)^2}{P\,G} = \frac{2025 + 18}{45} = 45{\cdot}4, \text{ exactly.}$$

A circular disc whose plane is perpendicular to the axis Q R, with G as centre, radius $= r$, and $G\,P = a$, will have its centre of oscillation at the same point O, as the cylinder; that is,

$$P\,O = \frac{a^2 + \frac{1}{2}r^2}{a} = a + \frac{r^2}{2a}.$$

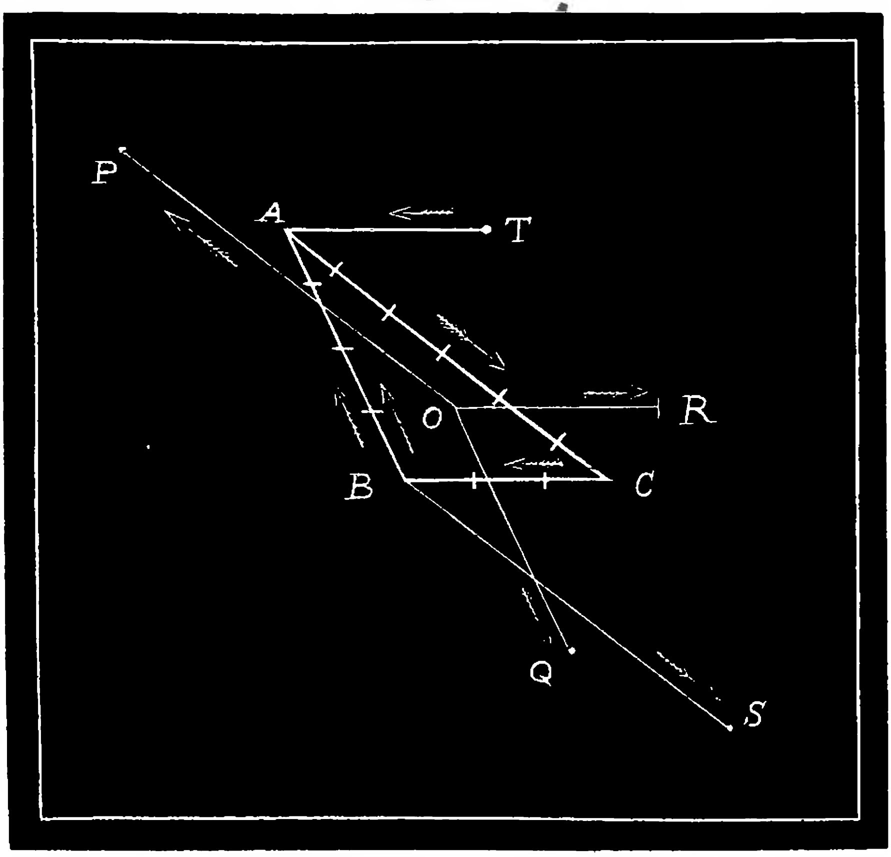
CHAPTER VII.

EQUILIBRIUM OF PRESSURES, THRUSTS, AND TENSIONS, FORCES, &c.

Anything which cannot be presented to the senses must be represented conventionally, or no idea of it can be entertained by the mind. We have no objection to represent the magnitudes and directions of forces, velocities, pressures, thrusts, and tensions by straight lines of different lengths and in different positions; but we object to the careless manner in which writers on mechanics represent forces, pressures, velocities, &c., by straight lines, triangles, and parallelograms, out of all proportion, and often, too, without the least regard to the action, direction, or nature of the force considered. In the sixth chapter we gave a conventional signification to the areas of plane figures, but in the present chapter the sides and angles of triangles, parallelograms, and other plane figures are employed to give an idea of mechanical operations and combinations.

The split arrow (⫽——→) is put to designate the directions of tensions, the arrow without feathers (——→) points out the directions of thrusts, while the complete arrow (⋙——→) applies to forces and velocities in general. If three forces, P, Q, R, Fig. 66, be represented by the abstract numbers, 6, 4, 3, respectively, then if A B C be a triangle whose side A C = 6, A B = 4, B C = 3, equal parts taken from any scale of equal parts; let P O be a tension equal and parallel to A C, Q O a tension equal and parallel

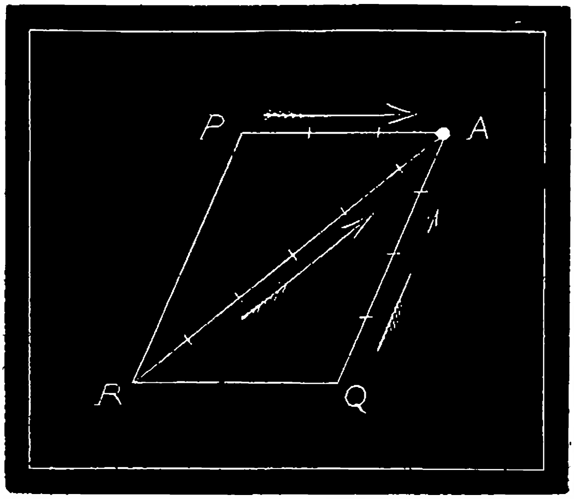
Fig. 66.



to A B, R O a tension equal and parallel to B C; these three tensions will keep the point O at rest. Three thrusts represented by three lines P O, O Q, O R, whose lengths are as 6, 4, and 3, respectively, and parallel to the sides of the triangle A B C, will also keep the point O at rest. Take A T equal and parallel to B C, then the two thrusts represented in magnitude and direction by the lines T A, B A, and the tension represented in magnitude and direction by the line A C, will keep the point A at rest. Again, if B S be equal and parallel to A C, tensions represented by A B, B S, and a thrust, C B, will keep the point B at rest.

Let P A, Q R, Fig. 67, be a parallelogram, if P A, Q A, represent the magnitudes and directions of two forces, two pressures, two thrusts, two tensions, or two velocities acting on the material point A; then the diagonal R A will represent the magnitude and direction of a single force, pressure, thrust, &c., equivalent to the operations represented by the lengths and directions of P A and Q A. Such imaginary parallelograms form a great portion of the stock-in-trade of writers on mechanics. We have introduced Figs. 66 to 70 to illustrate the manner in which the different forms of arrows are applied. The properties pointed out in Fig. 66 depend upon the truth asserted of the parallelogram P A Q R, Fig. 67, which we will prove presently.

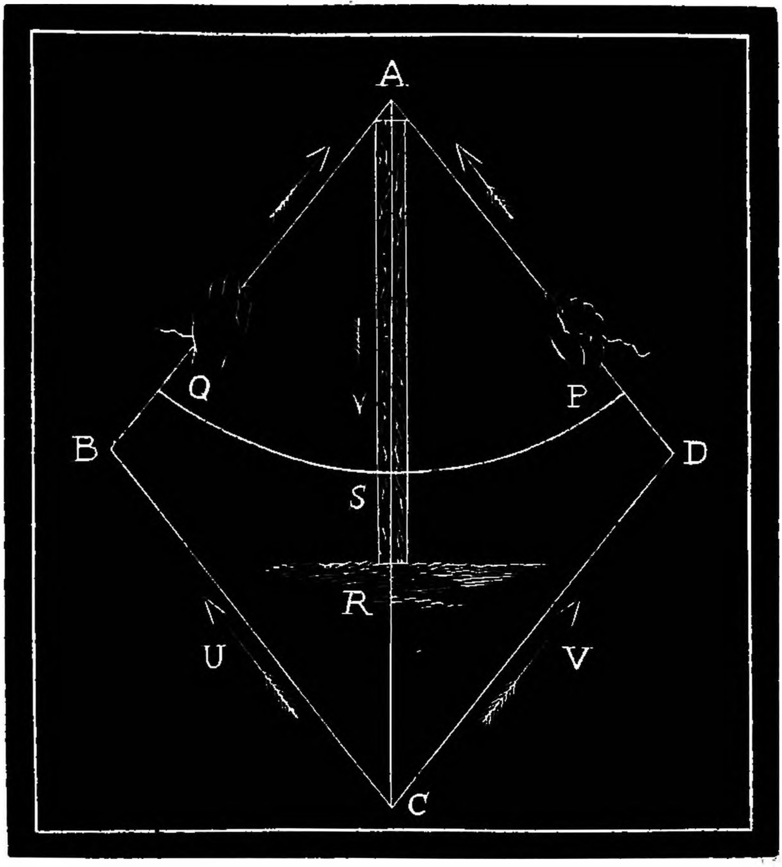
Fig. 67.



The principle of sufficient reason; first employed by Archimedes in demonstrating the fundamental propositions of mechanics.

Let P and Q, Fig. 68, be two equal forces acting in the same plane on the rigid perpendicular prop A R, the point R fixed; further, let these forces P, Q make equal angles, S A D, S A B, with the prop A R, which may turn to the right or left round the fixed point R. But as the forces P and Q and the angles B A S and D A S are also equal, there is no reason why the post A R will turn to the right or left; hence it will stand, and the two tensions will be neutralised by the perpendicular thrust of the post A R. This we call a mechanical demonstration, and the principle applied "the

Fig. 68.



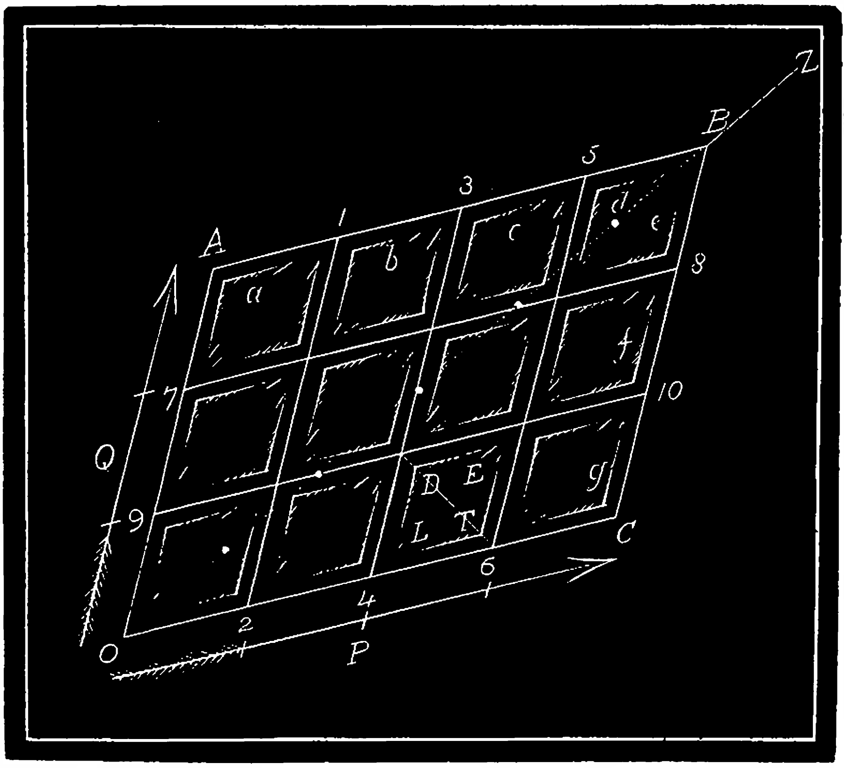
principle of sufficient reason." For there is no reason why the post under consideration should fall to one side more than another, hence we conclude it will fall to neither side. Let A B C D be a parallelogram whose sides are all equal; if the length of the line A B be taken to represent the magnitude of the force Q, then A D will truly represent the force P, and two other equal forces, V and U, applied to the point C, represented in magnitude and direction by the lines C D, C B, will counteract the forces P and Q the same as the post A R, the diagonal A C being rigid. The principle of sufficient reason applies here as in the former case. It is evident that any number of equal forces may be applied in the directions of the arrows, P = Q = V = U, without disturbing the form of the equal-sided parallelogram A B C D, or the position of the rigid diagonal A. The abstract truth of the parallelogram of forces, thrusts, tensions, pressures, or velocities may be established by reasoning as follows: Let A B C O be a parallelogram; when its sides can be expressed by numbers, it may be divided into small parallelograms, the sides of which are all equal to one another. We have selected a very simple case, and taken O C = 4 equal parts and A O = 3 of the same parts.

Let the force P be to the force Q as *m* to *n*; in the present example,

$$P : Q :: 4 : 3.$$

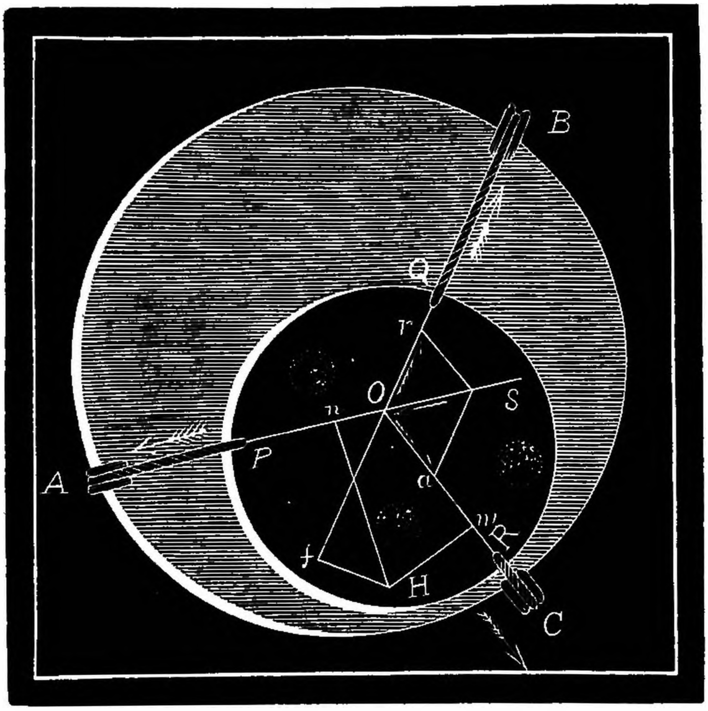
The forces P and Q applied to the material point O in the direction O C, O A, may evidently be represented by the lines O C, O A; if a new set of forces, represented by the small parallelograms, be introduced as represented by the arrows in the Figure, the combined action of the forces P and Q will not be interfered with, nor will the equilibrium of the Figure be disturbed. Take, for example, the

Fig. 69.



small equal-sided parallelogram D E T L, and for a moment suppose the diagonal D T rigid; then *the principle of sufficient reason* applies to the forces introduced, and no disturbance takes place. Let us now view these forces under another aspect. The forces along the sides of the small parallelograms on the line C O are equal to the force P, but in a contrary direction, and hence destroys it. The forces represented on the line 9, 10, also destroy one another. This may be said of the forces on the line 7, 8; but the forces on the line A B are not neutralised, and are exactly equal to the force P, but applied to the point B in the diagonal O B. Again, the forces along the sides of the small parallelograms on the line A O are equal to the force Q, but in a contrary direction, and hence may be said to destroy it. The forces pointed out by the arrows on the line 1, 2, also neutralise one another; the same may be said of the forces supposed to be introduced on the line 3, 4, and so on until we arrive at the forces pointed out on the last line B C, which, together, are exactly equal to the force Q, but applied to the point B in the diagonal O B. According as the equal parts be long or short, the point B will change its position; but it will always be in the diagonal O Z, and the forces $a+b+c+d=$P and $g+f+e=$Q will in all cases have their resultant position in the diagonal O Z. Had we supposed the force P to be divided into 1000 equal forces, O C would be represented by 1000 equal parts, and O A$=$Q would then be composed of 750 such parts, and our reasoning on the 750000 equilateral small parallelograms of forces would result in the same conclusion. The equilibrium of forces and pressures may be illustrated in a practical way by a simple mechanical contrivance, represented in Fig. 70, easily constructed. Suppose P Q R to be a slate or board rolling freely upon three equal spherical balls placed on a horizontal table, A B C. If any three points, P, Q, R, be taken in the surface of the plane movable on the balls, and cords, A P, B Q, C R, be attached, passing over the pulleys, with a different amount of weight attached to the other ends; the ends with the weights attached are not represented in the cut. The

Fig. 70.



board P Q R will roll upon the balls, and ultimately come to rest upon the table A B C, where the three forces destroy each other. When this experiment is carefully made, it will be found that the directions of the forces A P, B Q, C R, meet in the same point, O. Again, from any scale of equal parts take O r = the pounds in the force drawing the cord Q B, and from the same scale of equal parts lay off Oa = the pounds drawing the cord R C; then, if the parallelogram O r s a be constructed, the diagonal Os will be in the direction of the third force acting over the pulley A. It will be further found that the units of length in Os, the diagonal, will = the weight or force acting by the cord A P.

If O a = 22 equal parts, O r = 28, and Os = 26, all measured on the same scale of equal parts, then, if 7 lbs. be attached to the cord Q B, and $5\frac{1}{2}$ lbs. to the cord R C, a weight of $6\frac{1}{2}$ lbs. must be attached to cord A B to maintain this equilibrium. When forces are thus represented by a parallelogram, as O r s a, the forces represented by the sides are called the component forces, while the force represented by the diagonal is denominated the resultant. Another experiment may be instituted to determine the intensity of forces by this simple apparatus. Instead of three forces being applied to the board, let there be any number; and suppose H to be any point taken in the movable plane P Q R, and from H let fall perpendiculars, H n, H m, H f, &c., in the directions, or directions produced, of the forces; then the units of length, taken from any scale of equal parts, multiplied by the pounds in the corresponding force, will be what is termed the *moment* of that force tending to turn the board or movable plane upon the point H. This being done, the principle denominated *the principle of the equality of moments* may be verified by experiment. Here but three forces are applied, but any given number may be introduced; in the present case,

$$\text{H}f \times \text{B lbs.} + \text{H}m \times \text{C lbs.} = \text{H}n\ \text{A lbs.}$$

The student must not confound *the principle of the equality of moments*, just defined, with the intensity of forces usually considered by examining the motions they produce, or the spaces passed over when the motions are uniformly accelerated. It has been before shown that, generally,

$$(\text{Force}) = \frac{\text{W}}{g}\ \frac{v}{t} \qquad (\text{I.});$$

t being an extremely small portion of time, denominated an element, during which the velocity v is generated. The above formula shows that if, by observing the laws of motion, we know for each instant the value of the ratio $\frac{v}{t}$, we shall then have that of the corresponding effort designated (Force) in the above formula. Suppose we know, from experiment, that the motion is uniformly accelerated, we have, for the space S,

$$\text{S} = \tfrac{1}{2}\text{V}_1 \times \text{T}^2.$$

$$\text{In which } \text{V}_1 = \frac{\text{V}}{\text{T}} = \frac{v}{t} \text{ as in (I.)};$$

$$(\text{Force}) = \frac{\text{W}}{g}\ \frac{2\,\text{S}}{\text{T}^2} \qquad (\text{II.})$$

Ques. 56. Suppose a locomotive weighing 10 tons (22400 lbs.) runs, with a uniform accelerated motion, a distance of 154·4 feet in 4 seconds; required the force in lbs. capable of imparting this accelerated motion, the friction of the rail not being considered?

$$\text{From (II.) (Force)} = \frac{22400}{32\frac{1}{6}}\,\frac{2\times(154\cdot4)}{4^2} = 13440 \text{ lbs.}$$

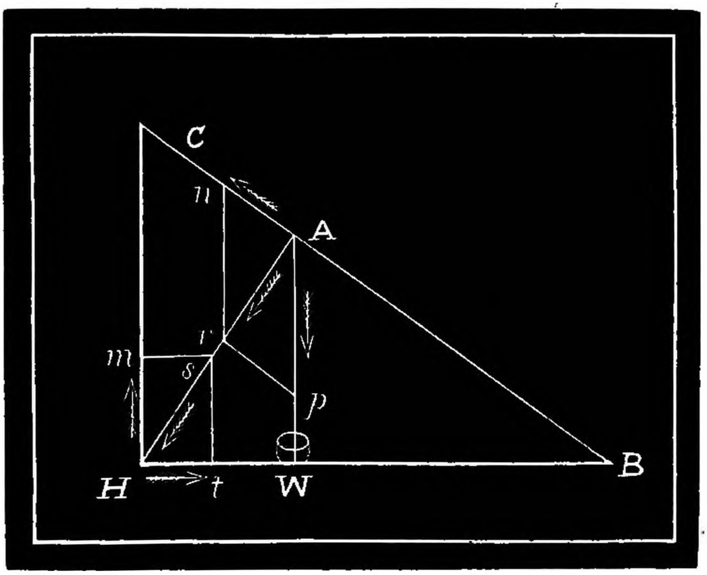
(See pp. 93, 94, 116.)

Definition.—A bar is called a *strut*, or a *tie*, according as a thrust or a pull is exerted along its line of resistance.

Ques. 57. Let C H B A be a *skeleton diagram* representing centres and lines of resistances, H B a platform with a load W = 47 cwt. upon it, C B a chain supporting both load and platform; find, by construction, the tension of the chain and the amount and direction of the pressure upon H, the point about which the platform turns, the weight of which being neglected?

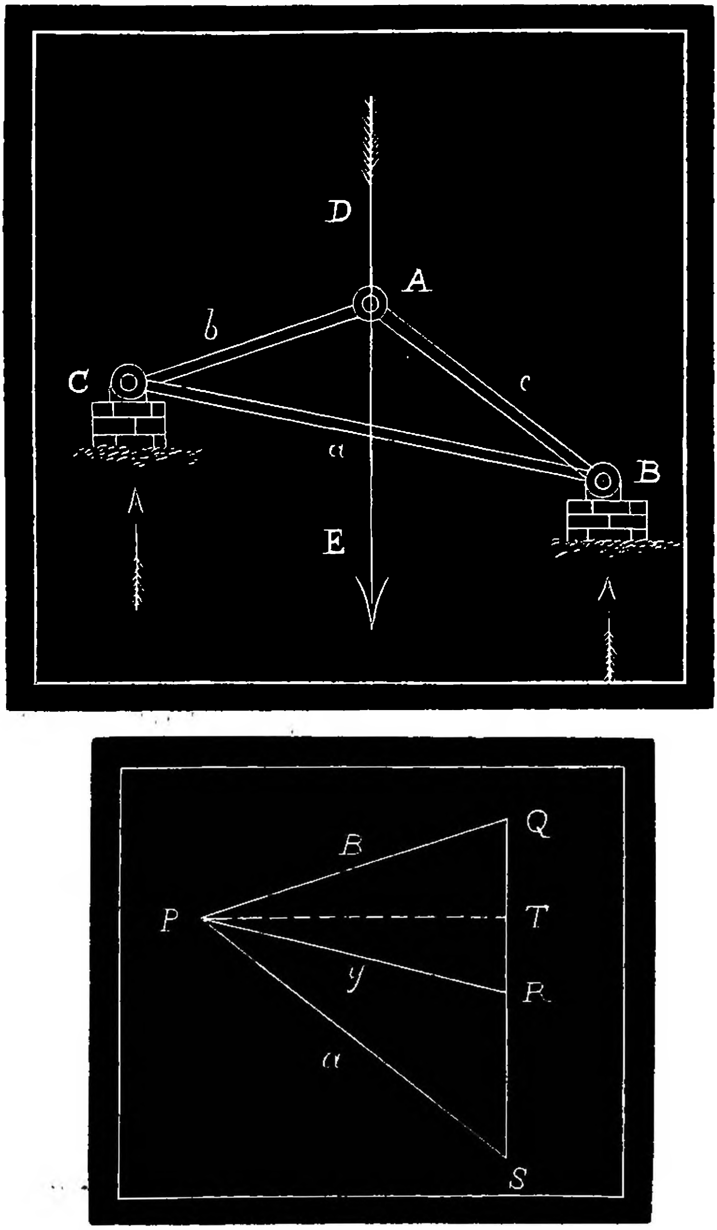
Draw W A perpendicular to H B, intersecting the chain in A; join A and H, and, from a scale of equal parts, lay off A p = 47, a part for each lb. in W which acts in the direction A W. Draw $p\,r$ parallel to C B, and $r\,n$ parallel to A W; the parallelogram A $n\,r\,p$ gives the required tension

Fig. 71.



and thrust. In the diagram A $r=37$ equal parts, answering to 37 cwt., the pressure on the hinge H; A $n=26$ equal parts, answering to 26 cwt., the pressure tending to break the chain. Take H $s=$ A r, draw $s\,m$ parallel to W H, and $s\,t$ parallel to H C; then H t gives the thrust of W against the wall which is represented by H C, and H m shows the pressure in the direction of the stationary wall H C.

Fig. 72.



Ques. 58. Given a triangular frame, A B C, Fig. 72, loaded and supported both in the direction of the vertical line D E; find the relative portions of the forces?

Put A, B, C for the three joints, and *a*, *b*, *c* for the three bars. These things being premised, from any point, P, draw P R, P Q, P S parallel to the lines of resistance *a*, *b*, *c* respectively; across these lines the vertical line Q R S, then the following proportions apply to the forces:

Load on A : supporting force at B : supporting force at C : stress along *c* : stress along *a* : stress along *b*; so stand in order and proportion the lines

Q S : S R : Q R : P S : P R : P Q.

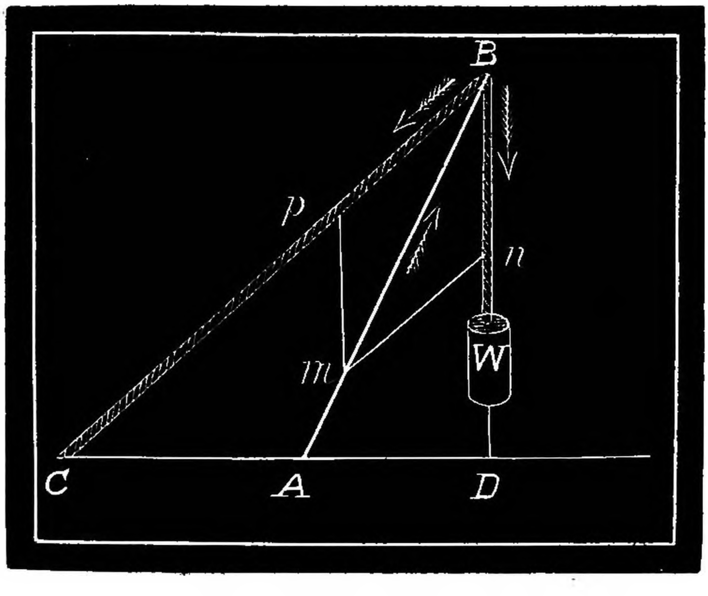
Out of the six proportions thus succinctly stated, to illustrate, we shall single out one:

Load on A : supporting force at B : : Q S : S R.

Ques. 59. Let a pole, A B, Fig. 73, be supported by a cord, C B, and carry a weight, W; required the tension of the cord, and the pressure on the pole?

From B to *n* lay off equal parts, equal to the units of weight in W; draw *m n* parallel to C B, intersecting B A in *m*; draw *m p* parallel to B *n*; therefore the units in B *p*

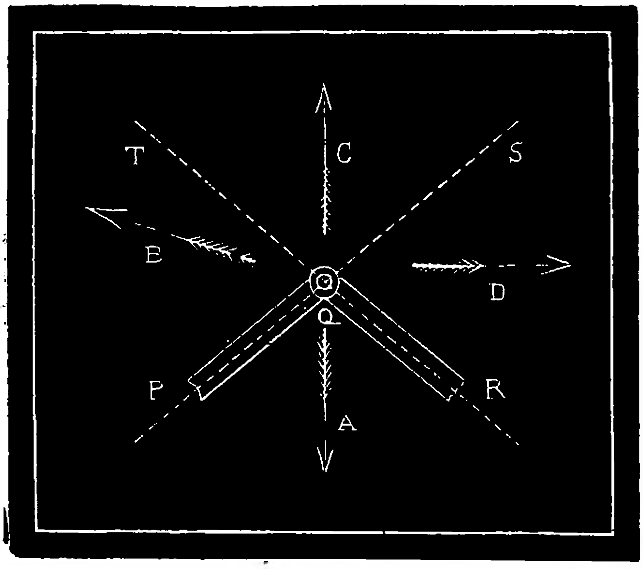
Fig. 73.



will give the tension of the cord C B, and the units, measured on the same scale of equal parts, in the diagonal B *m*, will give the thrust on the pole A B.

Struts may be distinguished from ties thus: In Fig. 74, P Q, Q R, represent two bars of a frame meeting at the joint Q, produce the lines of resistance beyond Q to S and T. Then a force in the direction of the arrow A, between the angle P Q R, both bars are struts; if the force be in the direction of the arrow B, in the angle T Q P, then P Q is a strut, and Q R a tie. Again, when the force is applied in the direction of the arrow C, both bars are ties; and, lastly, when the force is in the direction of the arrow D, in the angle S Q R, then P Q is a tie, and Q R is a strut.

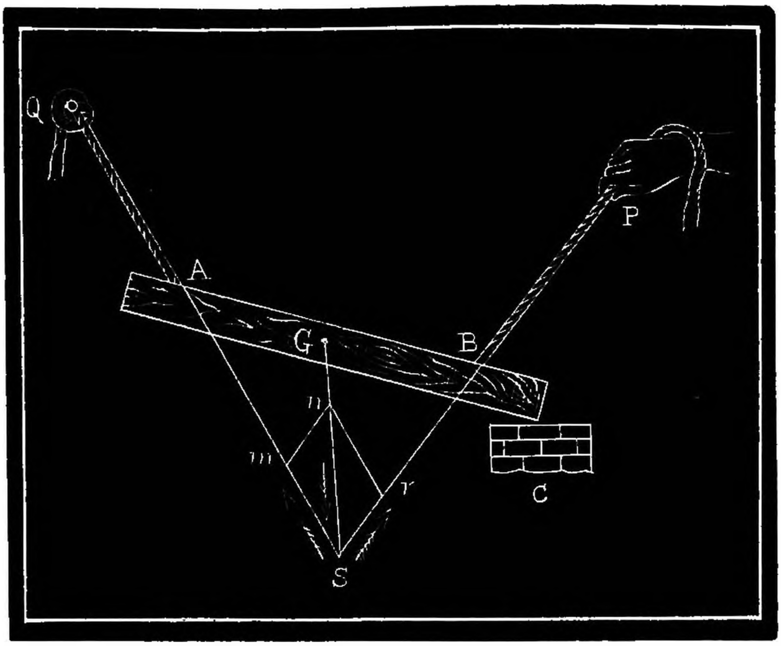
Fig. 74.



Ques. 60. A beam, A B, resting on a wall, C, and supported by a cord, Q A; it is required to determine the direction and tension of the cord B P, so that the beam may not change its position when the wall C is removed, the point B being also given?

Let G be the centre of gravity of the beam, through which draw the vertical line G *s* in the direction of the

Fig. 75.



plumb line, to meet Q A produced in *s*, join *s* B, and produce it; B P in the direction of *s* B shows the direction a cord fastened at B must take so that the beam will not change its position when the wall C is removed.

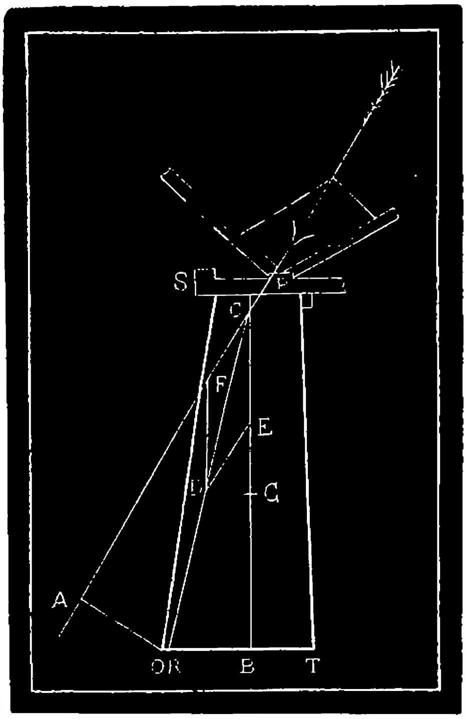
To find whether a pier, or other support, S T, Fig. 76, will overturn by the action of a force P, or the resultant of several forces operating in a given direction, P C A. Produce the direction of the force P, and let fall the perpendicular O A; then, in order that the structure may not turn on the point O, we must have, G being the centre of gravity,

Weight of the structure S T × O B greater than P × O A. When (weight S T) × O B = P × O A, the pier will be on the point of turning on the edge, O.

By Construction.

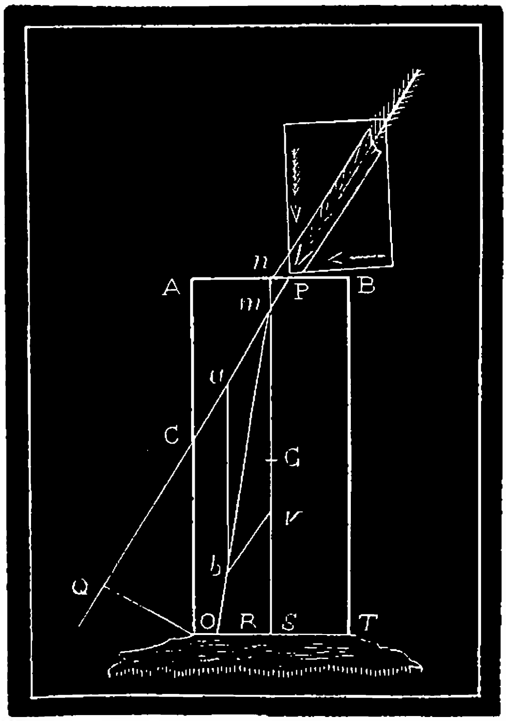
Let B C be the vertical line passing through the centre of gravity of the pier or pillar, intersecting the direc-

Fig. 76.



tion of the force P in the point C; from a scale of equal parts take C F = the units in the pressure P, and from the same scale take C E = the units in the weight of the structure, S T; complete the parallelogram of forces, C D E F, then C D will give the amount and direction of the single force tending to overturn the structure or support, S T. When C D produced intersect the base within the edge O, the structure will stand; but should the point of intersection, R, fall without the base, the structure must fall.

Fig. 77.



Ques. 61. A T, Fig. 77, is a pier, or buttress, weighing 860000 lbs., P a pressure of 320000 lbs.; A B = O T = 6·6 ft.; O A = B T = 15 ft.; the cross section A B T O is a rectangular parallelogram, of which G is the centre of gravity. The direction of the force P tending to overturn the structure cuts A B 4 ft. from A, and A O at C, 7 ft. from A. It is required to find whether the structure will stand or fall, and what will be the amount of the pressure, P, just sufficient to overturn the solid mass of which A B T O is a cross section?

$$CP=\sqrt{AC^2+AP^2}=\sqrt{7^2+4^2}=8{\cdot}0622577,$$
$$CP : PA :: CO : OQ,$$
$$\text{or, } 8{\cdot}0622577 : 4 :: 8 : 3{\cdot}969=OQ.$$

Since A T is a rectangular parallelogram, a vertical line, S G *m*, passing through the centre of gravity, G, divides the base O T into two equal parts.

$\therefore$ O S = S T = 3·3 ft.; then we can compare

Moment of the wall $=860000\times OS=2838000$.

Moment of the pressure $=320000\times OQ=1270080$,

2838000 being greater than 1270080.

The structure will stand.

Putting x for the pressure that will just overturn it, acting in the direction of P,

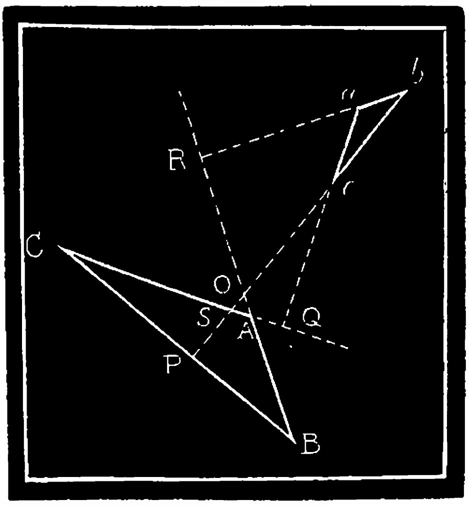
then, $x\times OQ=2838000$.

$$\therefore \quad x=\frac{2838000}{3{\cdot}9069}=715022 \text{ lbs.}$$

If P=715022 lbs., the point R will coincide with the point O on the outer edge of the structure.

Geometrical Proposition.

Fig. 78.



If the sides, or the sides produced, of one triangle, *a b c*, Fig. 78, be respectively perpendicular to the sides, or sides produced, of another, A B C, these triangles are similar.

a Q, *b* P, *b* R, represent the three perpendiculars,

Angle C + C S P = a right angle,

Angle c + c S Q = a right angle;

But angle C S P = angle c S Q,

$\therefore$ angle c = angle C.

In the same way the angle *b* may be shown to be equal to the angle B; and consequently the angle a=A.

Funicular Poylgon.

Funicular Polygon is the term employed to designate a polygon formed of rods, chains, or cords, whose angular points are solicited by any forces whatever. The equilibrium of such a system of thrusts, tensions, &c., are subject to rules easily determined. First, let us suppose that the polygon has not all its sides in the same plane, and is represented by the contour, A B C D E F. Let P, Q, R, S, T, . . . be the forces acting upon each of the summits A, B, C, D, E; the respective tensions of the sides A B, B C, C D, DE, E F, represented by T, T_1, T_2, T_3 . . respectively. If the entire polygon is in equilibrium, it must be so for all the summits separately, and since T_1 must be equal and opposite to the resultant of T and P, it follows that any two sides uniting at the same summit, and the direction of the force acting upon this summit, must be in the same plane; and the tension T_1 must be equal and opposite to the resultant of the force Q and tension T_2, and consequently in substituting at the summit C, for the tension T_1, its two components P and T, the forces P and Q and the tensions T and T_2, supposed to be transferred to the point C, parallel to themselves, must be in equilibrium. We see also that the forces P, Q, R, and the tensions T and T_3, supposed to be transmitted to the summit D in parallel

Fig. 79.

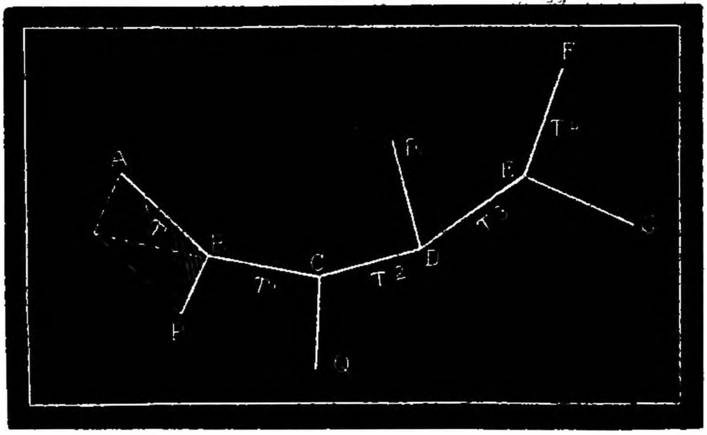
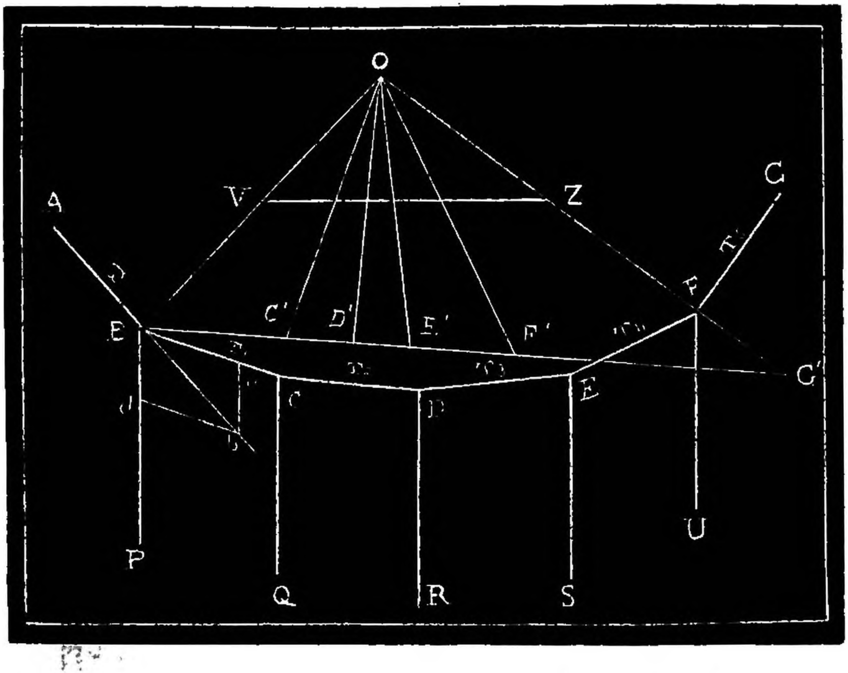


Fig. 80.

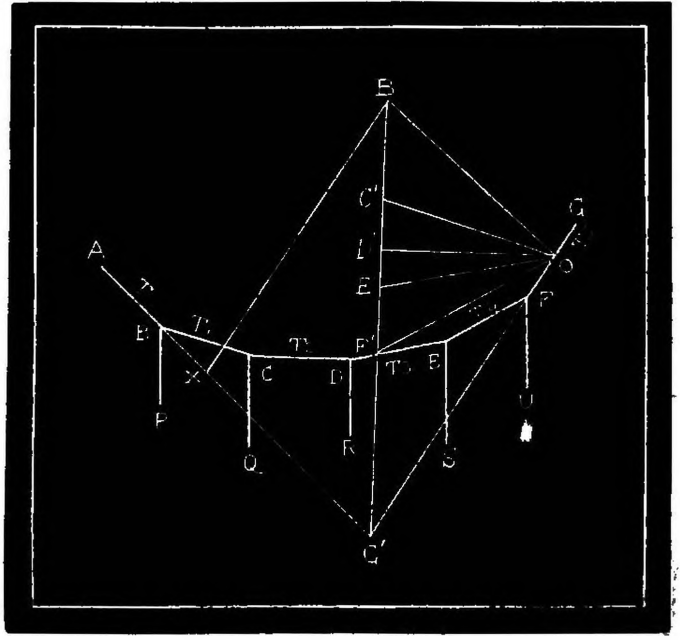


directions, must then produce an equilibrium. By continuing this process, we arrive at the conclusion, for the equilibrium of the funicular polygon, that when all the external forces and the tensions of the extreme sides are regarded as transferred parallel to themselves to any summit, they must necessarily produce an equilibrium there. These remarks are independent of the direction of the forces and the nature of the sides, and are applicable when the sides are subjected to efforts of compression instead of tension, with the reservation solely that the sides be sufficiently rigid to resist the compressions, without a change of form. Where the forces soliciting the funicular polygon are weights, in which case the forces P, Q, R, S . . are all vertical and parallel, and the polygon are necessarily in one plane; for, the direction of each force, and those of the two sides meeting at the summit, are in the same vertical plane; and, as through one side we can draw but one vertical plane, it follows that the two vertical planes, containing the same side of the polygon, are coincident, and so for the other sides. Again, since all the external forces P, Q, R, . . and the tensions T_1, T_4 of the first and last sides are in equilibrium, T and T_4 must also be in equilibrium with the single resultant of all the parallel forces.

Determination of the Tensions by a Graphical Construction.

If, in accordance with the preceding views, we describe upon the side A B, produced, from any convenient scale of equal parts, a length equal to the tension T, and construct the parallelogram B *a b d*, Fig. 80, the side B *a*, measured on the same scale of equal parts, will represent their tension, T_1 of the side B C, and the side B *d*, the external force P. Then, if we draw through the point B, for example, a line B G^1 perpendicular to the directions of the forces P, Q, R, . . . upon which lay off B C^1, C^1 D^1, D^1 E^1, E^1 F^1, F^1 G^1, proportional to the forces P, Q, R, S, U, and from the same point erect the line B O perpendicular to A B with a length proportioned to the tension T, or to B *b*, the triangle B O C^1 having two sides, B O, B C^1, respectively perpendicular and proportional to the sides B *b* and B *d* of the triangle B *b d*, must be similar to it by the geometrical proposition preceding. Then the third side,

Fig. 81.



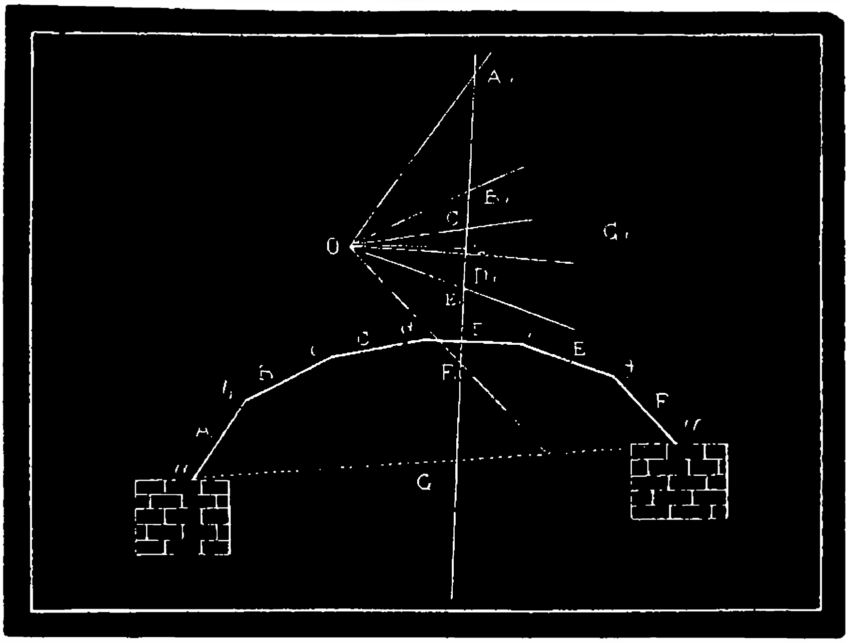
O C^1, of the first will be proportional to the third side, *b d* or B *a*, or to the tension T_1, and will be perpendicular to the side B C of the polygon; the following triangle will be in the same condition, in relation to the side C D, to the force Q, and the tension T_2, which will be proportional to the side O D^1, and so on. Then, if the weight soliciting the different summits of the contour A, B, C, . . . are in equilibrium, and we lay off upon a horizontal line lengths respectively proportional to these weights, and then from points of this line corresponding to each summit draw straight lines perpendicular to the directions of the sides of the polygon, all these right lines will intersect at the same point, and their lengths will be proportional to the tensions of the sides of the polygon, which will thus be determined. If a line V Z be drawn parallel to B G^1, the triangle V O Z answers the same purpose as the triangle B O G^1. The triangle B O G^1, Fig. 81, expresses the same relations as the triangle B O G^1, Fig. 80, B G^1 being taken equal to the sum and in the direction of the forces P, Q, R, . . . B O is drawn parallel to the direction of T, and B X parallel to the direction of T_5; G^1 X gives the tension of T, and G^1 O the tension of T_5, both measured on the same scale of equal parts as B G^1. O B is parallel to T or A B; O C^1 is drawn parallel to B C; O D^1 to C D; O E^1 to D E; O F^1 to E F. .

We have been very particular to pass nothing over that might make the proper relations and actions of forces, thus circumstanced, clear to the mind of the student, since the proper arrangement of thrusts and tensions, the action of braces, struts, and ties, in bridge-building, roofing, and in all sorts of lattice-work, designed to combine strength with lightness, depend upon the extension of the simple principles here presented.

If the same Fig. 81 be inverted, then such frame, Fig. 82, consists chiefly of struts, and is, therefore, unstable unless their ends are made fast by suitable stays. In a polygonal frame loaded and suspended vertically, represented by the skeleton diagram, Fig. 81, the bars which are struts in Fig. 82 become ties, and the frame is stable and yet flexible.

The diagram of forces for Fig. 82 may be constructed as follows:—Suppose the polygonal frame loaded vertically and supported vertically, let A, B, C, D, . . . be the bars; *a*, *b*, *c*, *d* . . . the joints, of which *b*, *c*, *d*, *e*, *f* are loaded, *a* and *g* are supported. Take any convenient point, as O, draw O A^1 parallel to A; O B^1 parallel to B; O C^1 parallel to C; O D^1 parallel to D; O E^1 parallel to E; O F^1 parallel to F; and O G^1 parallel to G. Then draw the

Fig. 82.



vertical line A^1 F^1 crossing the lines O A^1, O B^1, O C^1. . . Then if the whole load on the frame be represented by A^1 F^1, the parts into which A^1 F^1 is cut by the lines O A^1, O B^1, O C^1, . . . will represent the fractional parts of the load that must rest on each of the joints to secure equilibrium.

A_1 B_1 represents the part of the load to be applied at the joint *b*; B_1 C_1 the part to be applied at *c*; C_1 D_1 the part to be applied at *d*; and so on. The lengths of the lines O A_1, O B_1, O C_1, . . . represent the resistances along the lines A, B, C, . . . to which they are respectively parallel. The two parts F_1 G_1, G_1 A_1, into which F_1 A_1 is divided by the line O G_1 parallel to G represent the supporting forces at *a* and *g*, that is, F_1 G_1 represents the supporting force at

g, and $G_1 A_1$ the supporting force at a. The horizontal stress of the frame is represented by the length of the perpendicular $O G_2$ let fall from O on $F_1 A_1$. If the angles of the polygonal figure A B C D . . . be given, the angles of the diagram of forces $O A_1 F_1$ are also given; hence, when the length of any one of the lines in $O A_1 F_1$ is also given, the lengths of the other lines are readily found by plane trigonometry, to any required degree of accuracy.

If the skeleton diagram A B C D . . . , Fig. 82, represent an open frame, the bar G is omitted; in this case the stress along the outer bars, represented in the diagram of forces by the lines $O A_1$, $O F_1$, must be met by oblique forces as abutments; for vertical supports would not be sufficient. This frame, being, as before observed, composed of struts, is unstable, and must be connected with suitable stays.

Ques. 62. Let A B and B C be a roof resting on the side-walls A and C; a uniform section of this roof weighs 6000 lbs., the angle B A C=34°, B C A=44°; it is required to find the thrust on the points A, C, the roof being without a tie-beam; what would be the tension of a tie-beam, A C, or of a rod connecting the feet of the rafters; and also find the direction and amount of the pressure of the roof to overturn the side walls?

To construct the diagram of forces, draw O a parallel to A B; O c parallel to C B; and O b parallel to A C, Fig. 83.

Draw the vertical line $a\ b\ c = 6000$ equal parts to represent 6000 lbs.

Let W=6000 lbs. applied, suppose, at the centre of gravity, G, of the section.

$$\therefore \text{ angle } a\,O\,b = 34°;\ b\,O\,c = 44°.$$

$$O\,a\,b = 90 - 34 = 56°;\ O\,c\,b = 90 - 44 = 46°.$$

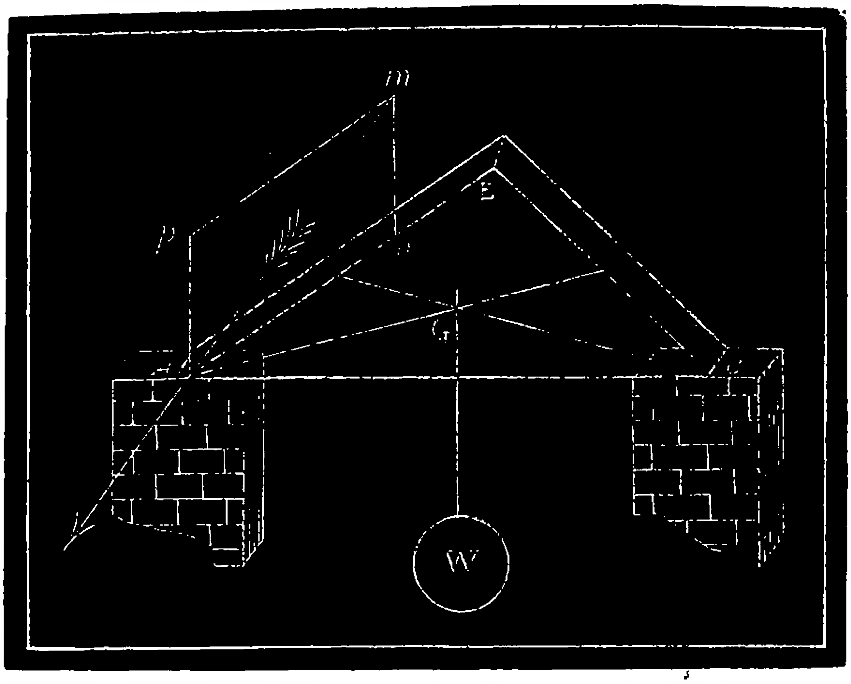
$$\text{Sin. } 78° : 6000 :: \text{sin. } 56° : O\,c = 5085{\cdot}35.$$

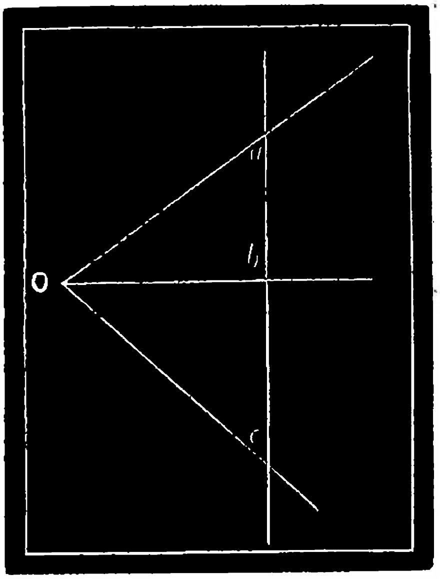
Natural sine of 78°=·9781476; natural sine of 56°=·8290376.

$$\text{Sin. } 78° : 6000 :: \text{sin. } 46° : O\,a = 4411{\cdot}44.$$

Natural sine of 46°=·7193398.

Fig. 83.





∴ The thrust on the point C=5085 lbs., and the thrust on the point A=4411 lbs.

As sine of 90° : O a : : sin. a O b : $a\,b$=2466·8.

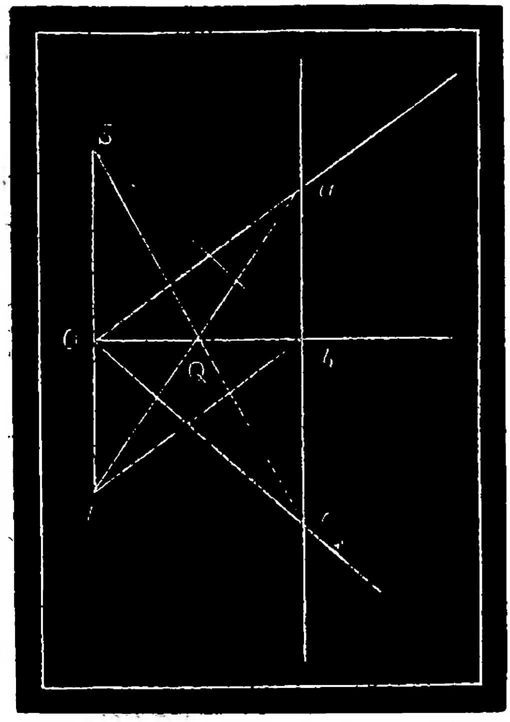
∴ $b\,c$=6000−2466·8=3533·2.

∴ The side wall at C supports 3533 lbs. of the weight W, and the side wall at A, 2467 lbs. nearly.

As sin. 90° : O *a* : : cos. *a* O *b* : O *b*=3657·3.

∴ The horizontal thrust along the side A C = 3657 lbs.

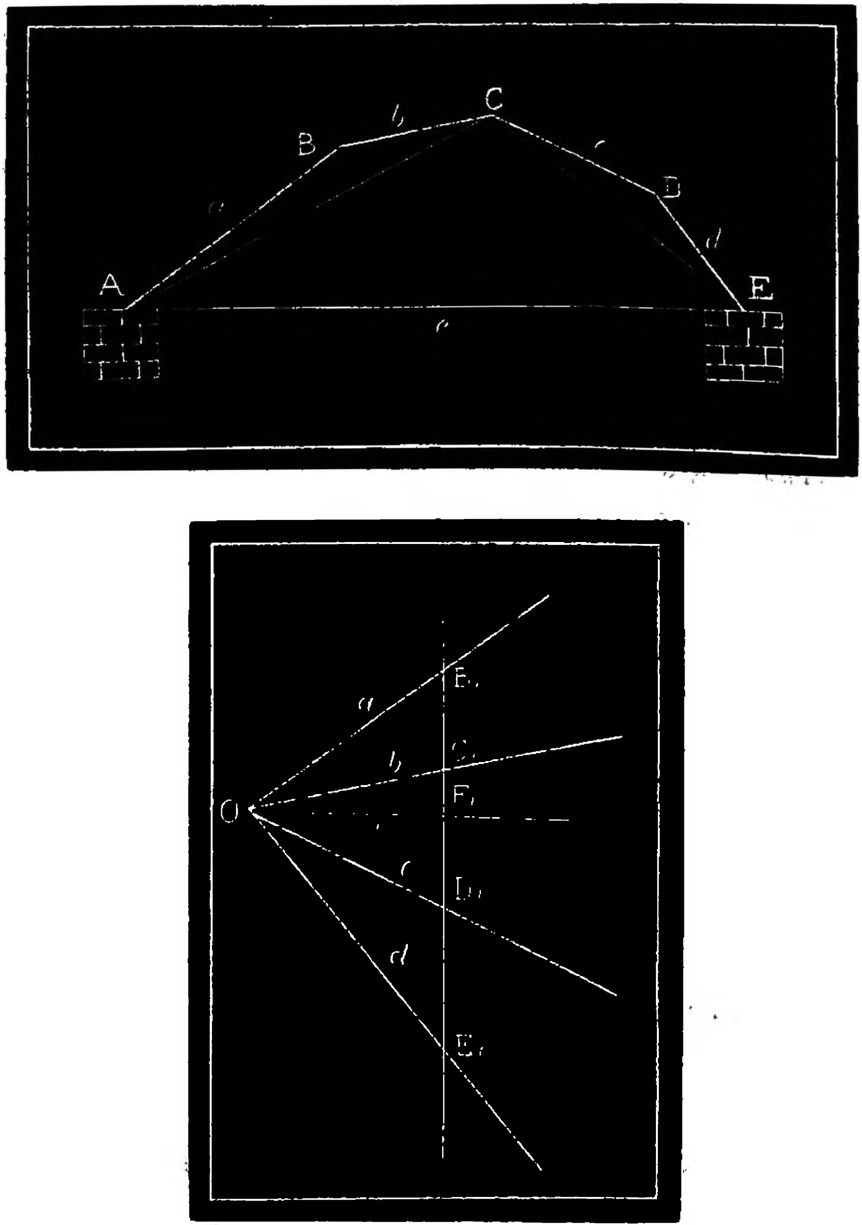
Fig. 84.



Draw A *p* perpendicular to A C, and = *a b*, the vertical weight at A, Fig. 83; make A *n* = O *a*, the thrust along the rafter A B; complete the parallelogram A *p m n*, and draw the diagonal *m* A, = the amount and direction of the pressure of the roof tending to overturn the wall A. Or, to avoid complicating the figure, we reproduce the diagram of forces, Fig. 84; taking the lines O *a*, *a b*, complete the parallelogram O *r b a*, then the diagonal *a r* gives the amount and direction of the pressure as well as the parallelogram A *n m p*, Fig. 83. On the lines O *c*, *c b*, construct the parallelogram O *s b c*, draw S *c* = the amount and direction of the pressure of the roof applied to overturn the wall C, when the tie A C is omitted.

Given the skeleton diagram A B C D E, Fig. 85, of a frame loaded and supported vertically; A E being horizontal. O B_1 is parallel to A B; O C_1 parallel to B C; O D_1 parallel to C D; O E_1 parallel to D E; and O F_1 is drawn parallel to A E; B_1 E_1 represents the whole load, and, being vertical, is perpendicular to O F_1. Now suppose

Fig. 85.



the angle $F_1 O C_1 = 12^\circ$, the natural tangent of which is ·21255 = $F_1 C_1$, when $O F_1 = 1$; the secant of this angle is 1·0223 = $O C_1$. $F_1 O B_1 = 37^\circ$; $F_1 O D_1 = 25^\circ$; $F_1 O E_1 = 50^\circ$; then the comparative lengths of the lines in the diagram of forces will stand thus:

	Tangent.	Secant.
$O F_1 = 1$,	$F_1 C_1$ = ·21255	$O C_1$ = 1·0223
	$F_1 B_1$ = ·75355	$O B_1$ = 1·2521
	$F_1 D_1$ = ·46630	$O D_1$ = 1·1033
	$F_1 E_1$ = 1·1917	$O E_1$ = 1·5557

This frame is vertically loaded with 5000 lbs.; how is it distributed, and what are the particulars of its action when the forces balance each other?

$F_1 B_1 + F_1 E_1$ = 1·94525 = 5000 lbs.

1·94525 : 5000 :: 1·0223 : 2628 lbs. stress along the bar *b* or B C.

1·94525 : 5000 :: 1·2521 : 3218 lbs. stress along A B or bar *a*.

1·94525 : 5000 :: 1·1033 : 2836 lbs. stress along bar *c*.

1·94525 : 5000 : 1·5557 : 3999 lbs. stress along *d*.

$F_1 B_1 - F_1 C_1 = C_1 B_1$ = ·75355 − ·21255 = ·541

1·94525 : 5000 :: ·541 : 1391 lbs. load on the joint B.

$F_1 C_1 + F_1 D_1 = C_1 D_1$ = ·67885

1·94525 : 5000 :: ·67885 : 1745 lbs. load on the joint C.

$F_1 E_1 - F_1 D_1$ = ·7254

1·94525 : 5000 :: ·7254 : 1864 lbs. load on the joint D.

1·94525 : 5000 :: $F_1 E_1$: 3063 lbs.,

the vertical weight falling on the support at E, the remainder of the weight, or 5000 − 3063 = 1937 lbs., must fall on the support at A. The horizontal stress being represented in the diagram of forces by $O F_1 = 1$, we have

1·94525 : 5000 : : 1 : 2570 lbs.,

the horizontal stress of the frame. When the load is properly distributed, the necessary stays, C E, A C, require but little strength; without stays such a frame, however well balanced, would be unstable. Here, as in other places, we have been careful to avoid encumbering the mind of the student with theoretical developments of a complicated nature. However, when these simple constructions and ordinary calculations are clearly understood, the system is easily generalised, and in practice readily applied to a variety of complicated structures.

PRACTICAL EXAMPLES OF WOODEN AND IRON BRIDGES IN WHICH THE ACTIONS OF FORCES EXPLAINED IN THIS CHAPTER HAVE BEEN APPLIED WITH MORE OR LESS SUCCESS.

Analysis of the Wooden Bridges of the United States and Canada.

The McCallum Inflexible Arched Truss Bridge approaches nearer the standard of perfection than any other wooden bridges that have fallen under the author's notice; this fact is easily established by comparison and demonstration.

Fig. 86.

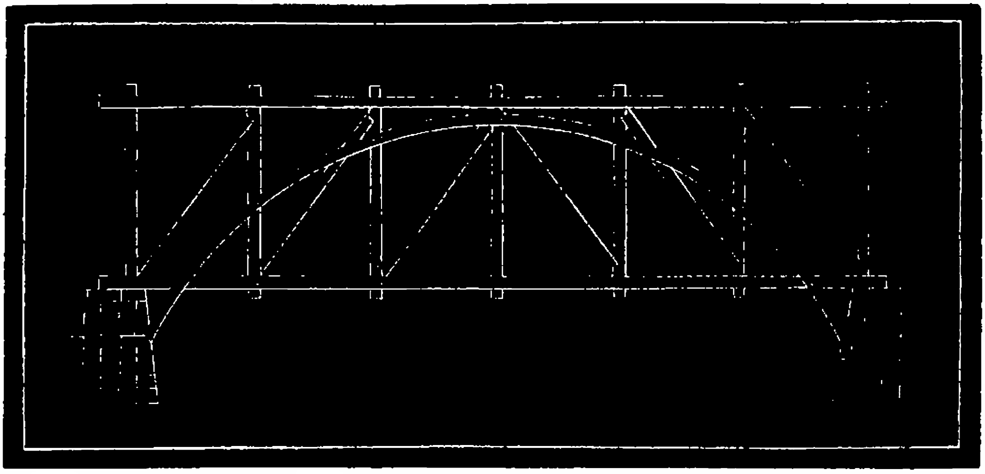


Fig. 86 is what is known as the *Burr Bridge.* It is composed of lower and upper chords, and posts and braces. The posts are framed into the chords, and the braces are framed into the posts. Arches are placed on each side of the truss, securely fastened thereto, and, extending below the lower chords, abut against the masonry.

This form of truss was extensively used throughout the United States previous to the introduction of railroads. Many spans were of great length, and in cases where the arches were large, and the masonry sufficiently permanent, this bridge was comparatively successful. Much difficulty was, however, experienced, by reason of the absence of counter-braces. A moving load produced a vibratory and undulating motion, tending to loosen the connexion of the timbers, which generally resulted in failure.

Many of the *first* railroad bridges, both in Europe and America, were built upon this plan; but much greater difficulty was found in adapting it to the use of railroads, than had been previously experienced in its use upon common roads. This difficulty arose from, 1st, the practical impossibility of perfectly combining the action of the *arch* and the *truss* (each system, of itself, being insufficient to carry the whole load); and, 2nd, the absence of counter-braces. These defects, clearly apparent in their use on common roads, were greatly aggravated under the increased and concentrated nature of the weight, and the rapid transit of trains on railroads. It is true, they were obviated in part by adding largely to the amount of material in the structures; but as the difficulty was inherent in the plan, violent contortions in shape could not be prevented, and these in time caused failures.

These remarks are intended to apply to spans of considerable length, as experience has proved that plans of even an inferior grade may be measurably successful in spans of ordinary length; whereas, nothing short of the most judicious distribution of material will ensure permanency in cases where long spans are indispensable, and any arrangement which can be made permanent in the latter case must certainly prove so in the former.

It is worthy of remark here, that this particular combination of the arch with the truss is even now, with some, a favourite idea, but it is believed that its warmest advocates will be generally found among those whose opportunities for practical investigation have been limited, and that it is only necessary that the question be properly presented to them, to produce a change of views in respect to it.

This partiality for the combination of the arch and the truss is attributable partly to the fact, that the simple truss has in many instances failed, and, as a last resort, the arch has been added, of such dimensions and strength as to be competent to carry the truss and load; the truss serving only as a stiffener to the arch, while the latter, thrusting upon the masonry, has sustained the whole weight. Besides, to the casual observer who has never studied bridge construction, this combination presents at least *an appearance* of great strength and solidity, which do not in fact exist.

That the simple truss without the arch has failed in some instances is unquestionably true; but while many of these failures have been caused from inattention to, or ignorance of, the laws regulating the composition and resolution of forces, by far the greater number have arisen from the inferior quality, or lack of the requisite amount of material, or from inferior workmanship.

The acknowledged failure of the *Burr Truss*, as applied to railroad purposes, led to the invention of several other plans, all of which were based upon the abandonment of the arch, and were aimed at perfecting a truss, which, *of itself*, would be sufficient to meet the emergencies of the case. This was in pursuance of what was considered a very reasonable hypothesis, namely, that *one* system properly proportioned must prove much superior to any method or arrangement in which the attempt was made to combine *two* distinct principles, in their nature heterogeneous.

Among the most prominent plans presented to remedy existing defects was one invented by Col. Stephen H. Long. This plan of bridge was composed of lower and upper chords, posts and braces, similar in outline and general arrangements to the Burr Truss, but differing from it in detail. An efficient system of counter-braces was introduced; these were made adjustable by wooden wedges, as were also the sustaining braces, by means of which any desirable elevation or deflection might be given to the truss. This plan of truss was rigid to a degree not previously attained; and to such an extent was this true, that, when properly adjusted, no perceptible deflection was produced by the passage of the load.

It was, however, found difficult to keep it in adjustment, in consequence of the great shrinkage of the wedges and other timbers of the truss.

The invention of what is known as the *Howe Bridge* followed. In this, as in Col. Long's bridge, the idea of combining the arch with the truss was *originally* abandoned for reasons heretofore given, and it was believed that this simple form of truss would prove equal to any reasonable requirement.

In the *Howe Bridge*, the posts used in the Burr and Long bridges are dispensed with, and iron rods substituted, by means of which any desirable *camber* may be given to the truss, thus overcoming the practical difficulty previously experienced in the adjustment of Col. Long's bridge, by the use of wooden wedges.

This method of producing *camber* is certainly an improvement upon the means adopted in the Long bridge for that purpose, but is much inferior to the latter in its method of counter-bracing, in that they are not adjustable, and perform a *negative* rather than a *positive* duty.

The *Howe Bridge* is composed of lower and upper chords, braces and counter-braces, vertical rods, and cast-iron *bearing blocks*. The braces abut the *bearing blocks*, which pass through the chords in such a manner as to permit the rods to bear directly upon them.

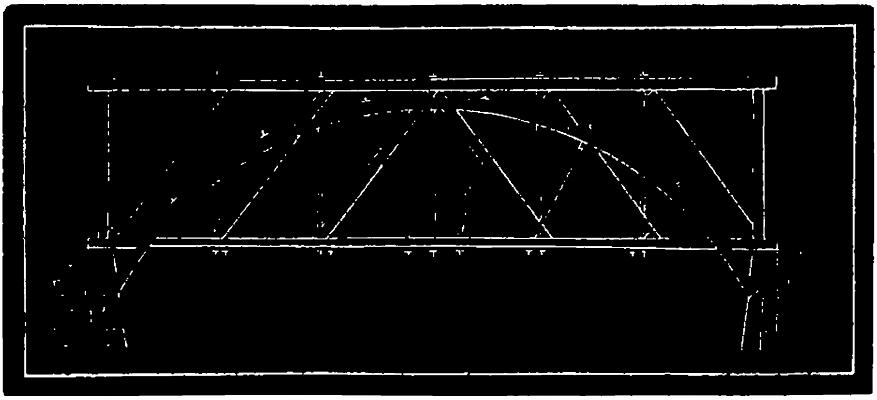
Spans of considerable length were built upon this plan, but experience proved that even this truss—like all others—had its limit, beyond which it could not be safely extended.

In the progress of railroad enterprises, in order to save large expenditures of money for masonry, longer spans than had been previously used became desirable, and in certain locations absolutely indispensable; besides this, locomotives were largely increased in weight, to meet the demands of traffic, and furnish a more economical mode of working; and thus arose the necessity for the adoption of some other expedient to meet the increased requirements of bridges. As all had been done by way of improving this truss that mechanical skill could devise, and which an extensive practice had amply afforded, it became evident that some *radical* change must be made in its arrangement, to enable it to meet the exigencies of the case.

In this emergency the *arch*, heretofore condemned in the *Burr Truss*, was again resorted to; for it had been proved, from the experience which its use in that truss had afforded, that an *arch* of sufficient size, abutting against permanent masonry, would place the truss in a position of secondary importance.

It will be observed that the arch of the Burr Bridge, Fig. 86, abuts upon the masonry in precisely the same manner as the arch of what is denominated the *Improved Howe Truss*, Fig. 87, and the difference between the two consists simply in the mode of connexion with the truss, and not in any change of principle or method of action.

Fig. 87.



It will be seen that the Burr arch is securely fastened to the posts and braces of the truss, forming a solid adjustable mass. In Fig. 87 the arches are not fastened to the braces or rods, but have an independent connexion with the lower chord of the truss, by means of rods radiating from the former to the latter. By this method it was supposed that any desirable adjustment could be effected, and that the strain could be put upon either system, or equally upon each.

This new arrangement, although plausible in theory, is found impossible in practice, for the following reasons:

1st. The rods from the arch to the lower chord are of various lengths, consequently the contractions and expansions must vary proportionately.

2nd. Not a single rod in the arch is of the same length as those in the truss, hence the expansion and contraction of the rods in the truss will vary from that in each, and all the rods connecting the arch with the lower chord.

3rd. This combination is exceedingly liable to maltreatment from the careless or ignorant.

4th. And even if it were everything in practice that is claimed for it in theory (which is not the fact), it involves a constant expenditure for adjustment, which must continue during the existence of the bridge itself.

The Burr Truss, Fig. 86, with all its defects, *can be made superior by far* to the *Improved Howe Truss*, Fig. 87. For, in the former, there may sometimes be a yielding and compression between the parts of the truss and those of the arch, producing a certain degree of united action; while in the Howe Truss everything depends upon the length of the rods, which must always change with the temperature, and thus render an approach even to perfect adjustment a matter of extreme delicacy.

But in either Fig. 86 or Fig. 87, it is clearly evident that, in order to have a structure absolutely safe, the arch and the truss—each of itself, independently of the other—should be of sufficient strength to sustain the whole load, that the strain may be borne alternately by each separate system.

In order to simplify and make clear the real points of difference existing in the combinations of the various plans of trusses of the same general outline, it may be stated that the material composing *any bridge truss*, whether of wood or iron, or of both, is subjected either to *tension or thrust*, and it is upon the proper application of these elements, together with a judicious distribution of the material, rather than upon any difference in detail that the perfection of any bridge structure depends; this may be illustrated by reference to Figs. 88, 89, and 90.

Fig. 88.

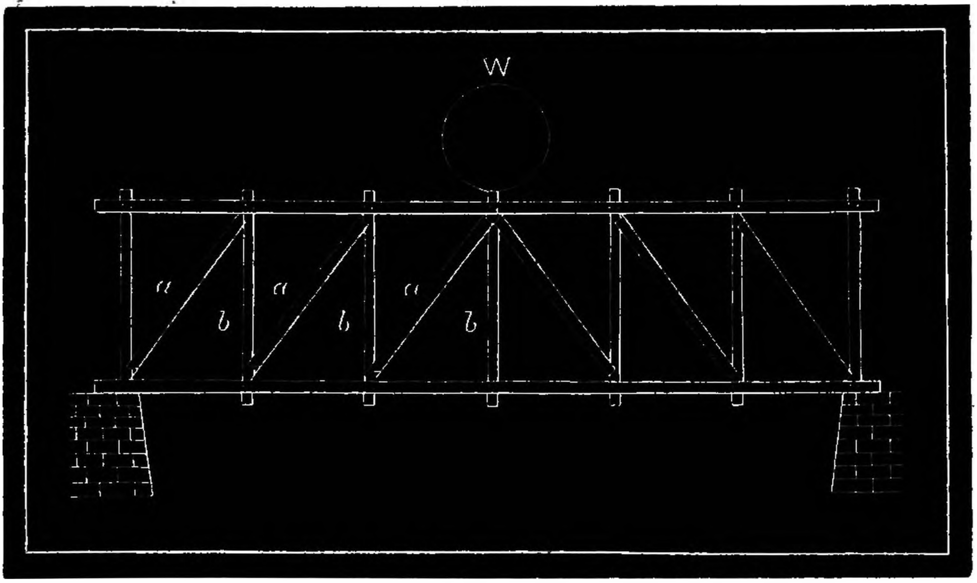
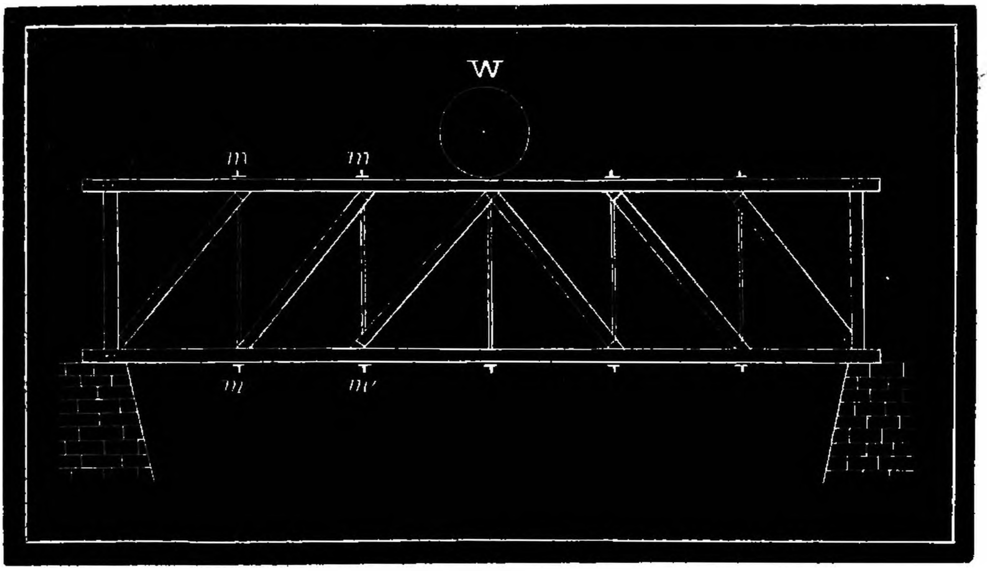


Fig. 88 is the truss of the Burr Bridge; in this the upper chord and braces are acted upon by thrust, and the lower chord and posts by tension.

Fig. 89 is the Howe Truss, without the counter braces;

Fig. 89.



in this also the upper chord and braces are subjected to thrust, and the lower chord and vertical rods are acted upon by tension.

Fig. 90.

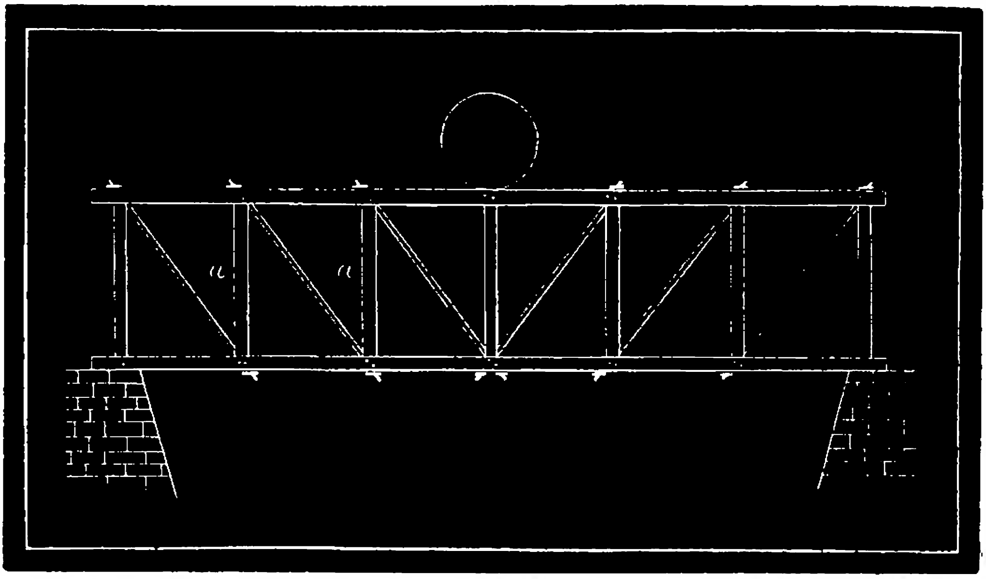
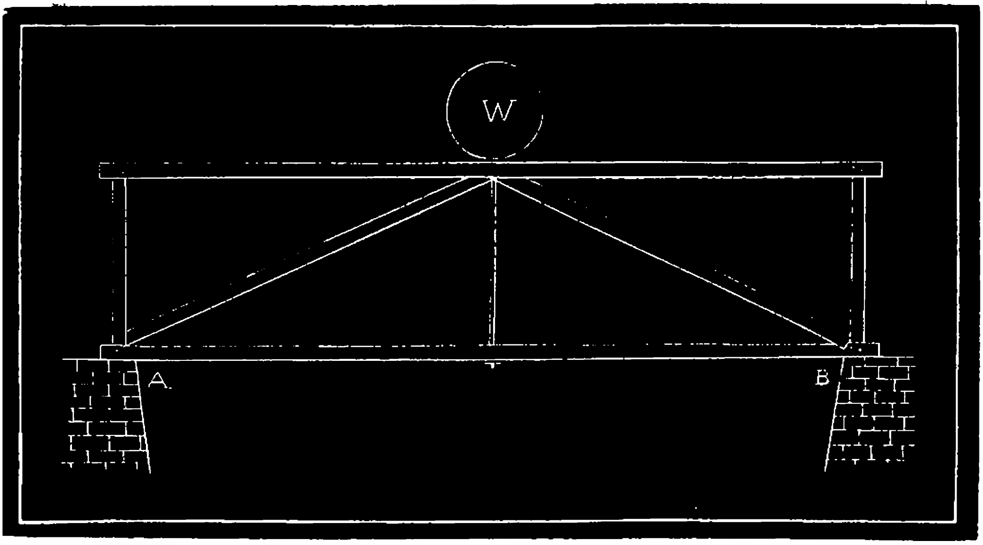


Fig. 90 is a plan of truss sometimes used, the counter rods being omitted; in this the upper chord and vertical struts are subjected to thrusts, and the lower chords and diagonal rods are acted upon by tension.

Fig. 91.



Upon a comparison of these plans it will be discovered that the variations between the Burr Truss, Fig. 88, and the Howe Truss, Fig. 89, consists in the use of vertical rods and bearing blocks in the latter, instead of vertical posts in the former, both having precisely the same duty to perform.

It will also be seen that Fig. 90 varies from Fig. 89, in that the rods are placed diagonally instead of vertically, changing the element of thrust from the diagonal braces in the latter to the vertical struts in the former, and transferring the element of tension from the vertical to the diagonal line.

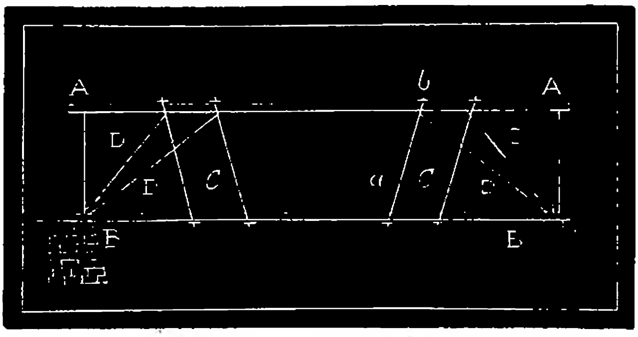
Much importance is sometimes attached to just such modifications in detail as exist in Figs. 88, 89, 90, *while the nature and intensity of the destroying forces are the same and equal in each.*

This has been proved by actual experiment, by the celebrated engineer and bridge builder, D. C. McCallum, as follows: Models were built, one on each plan, of equal length and height of trusses, containing the same sectional area and kind of material in chords and braces, and of equal perfection in details and workmanship, when it was found that the real difference in strength was unappreciable, and it may be well to add, that any given amount placed upon each, in progress of the experiments, presented precisely the same characteristics and contortions in shape, until final failure took place.

All bridges having their chords parallel, irrespective of the particular method adopted in combining them, and regardless of the amount of material used in their construction, when loaded to nearly the point of fracture, present somewhat the same appearance, the greatest deflection *being invariably* at points near the abutments. This will be understood by the statement, that the vertical strain is increased, as the distance from the centre, to the ends of the truss; at the centre the vertical strain is nothing, and at each end of the truss it is equal to one-half the weight of the structure and its load.

In point of strength, the arrangements, Figs. 88, 89, 90, are not superior to the simple combination, Fig. 91.

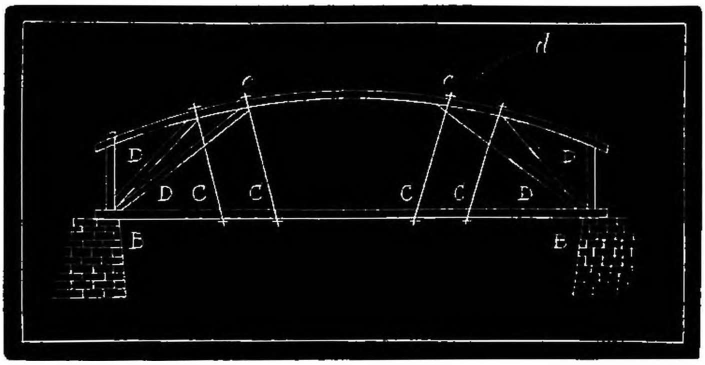
Fig. 92.



All bridges having their chords parallel exhibit the same uniformity of action, and may be illustrated by reference to Fig. 92, in which A A is the upper chord; B B, the lower chord; C C, tension rods; D D, braces.

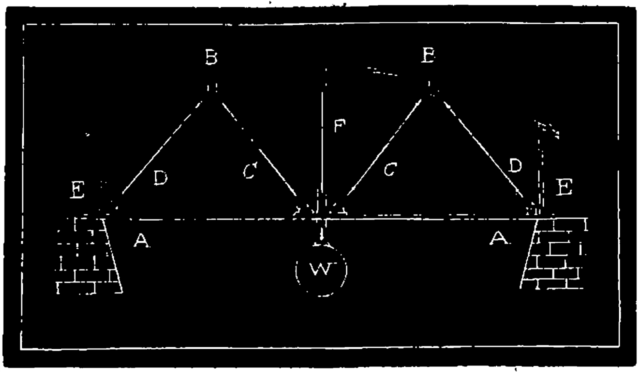
When a sufficient weight is applied to any truss of this outline, to cause deflection below the straight line, the upper ends of the braces, D D, are made to approach each other, and the distance between the ends is diminished, and as the deflection increases, the upper ends of the braces, D D, will describe arcs, *a b*, of a circle *downwards*, the radius of which being the length of the braces, D D. But when the upper chord is arched, as in Fig. 93, a sufficient weight will cause the braces, D D, to describe an arc *upwards*, represented by *c d*, Fig. 93. When the chord, *c e*, becomes straight, the arc will then be described downwards, as

Fig. 93.



shown in Fig. 92. As an illustration of the McCallum *Inflexible Arched Truss*, see Fig. 94, in which A A is lower chord; B B, upper chord; C C, tension rods; D D, braces; E E, struts; and W, weight.

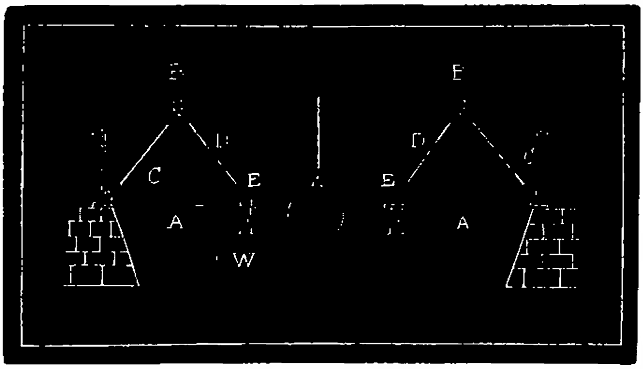
Fig. 94.



Upon an inspection of this figure, it will be seen that any deflection produced upon the centre of the arch, by means of the weight W, will cause the points B B to separate, by thrusting *outward*, and in the direction of the ends of the truss, producing an *upward* movement of the upper chord, at the ends of the braces D D, the latter describing arcs of a circle *upward*, and from thence will be communicated, by means of the tension rods C C, to the centre of the lower chord, raising the latter at the point where the rods, C C, meet.

By removing the weight W, and inserting a vertical strut at F, the upward movement of the chords will be arrested by the weight W. This peculiar action may be described as follows:

Fig. 95.



Any deflection produced in the centre of the arch will cause an *outward* and, consequently, an *upward* force at the upper ends of the braces, which, by means of the tension rods and strut, is transferred directly back to the under side of the arch, producing an *upward* force at the latter point, equal to the original *downward* force applied on top of the same.

This combination of forces is in agreement with a well-known law, namely, when two forces of equal powers of resistance are opposed to each other, a state of rest is produced.

For a further illustration of the action of this truss, see Fig. 95, in which A A are pieces of the lower chord, the centre being removed; B B, upper chord, deflected by the weight W. C C are braces which pass through the lower chord and rest upon the masonry. D D are tension rods. It will be seen that the ends of the pieces of lower chord at E E are raised considerably above a horizontal line. This upward tendency will continue until the upper chord between B B is deflected below a straight line, when the action will be reversed.

Fig. 96.

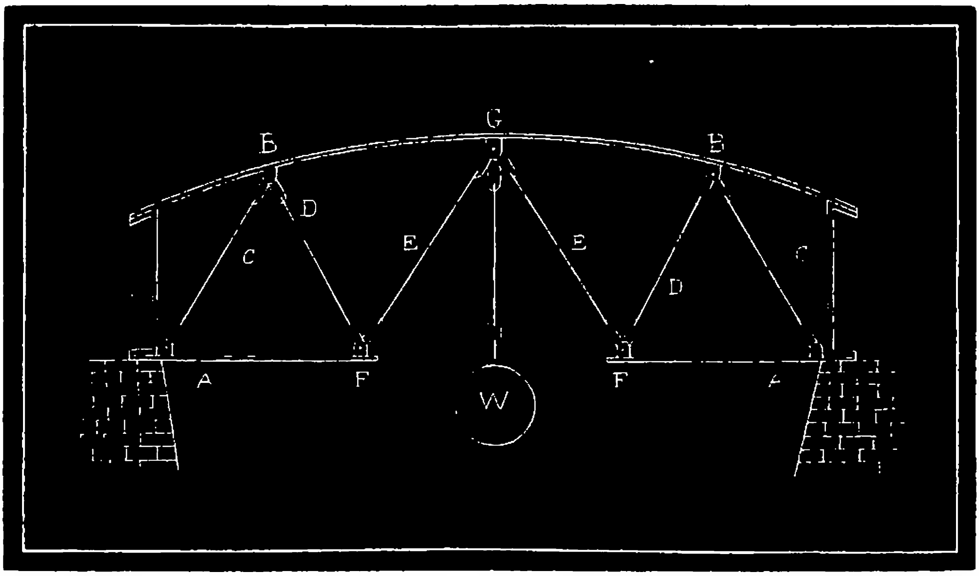


Fig. 96 exhibits the forces at a state of rest, in which A A are portions of the lower chord; B B, upper chord; C C, arch braces, which pass through the lower chord, and rest in the masonry; D D, tension rods; E E, braces; W, weight.

It will be seen that the strain produced by the weight W is transferred to the lower chord by means of thrust upon the braces E E, to the points F F, and, by means of tension on the rods D D, to the points B B, and from thence it is brought upon the arch braces C C, which rest upon the masonry.

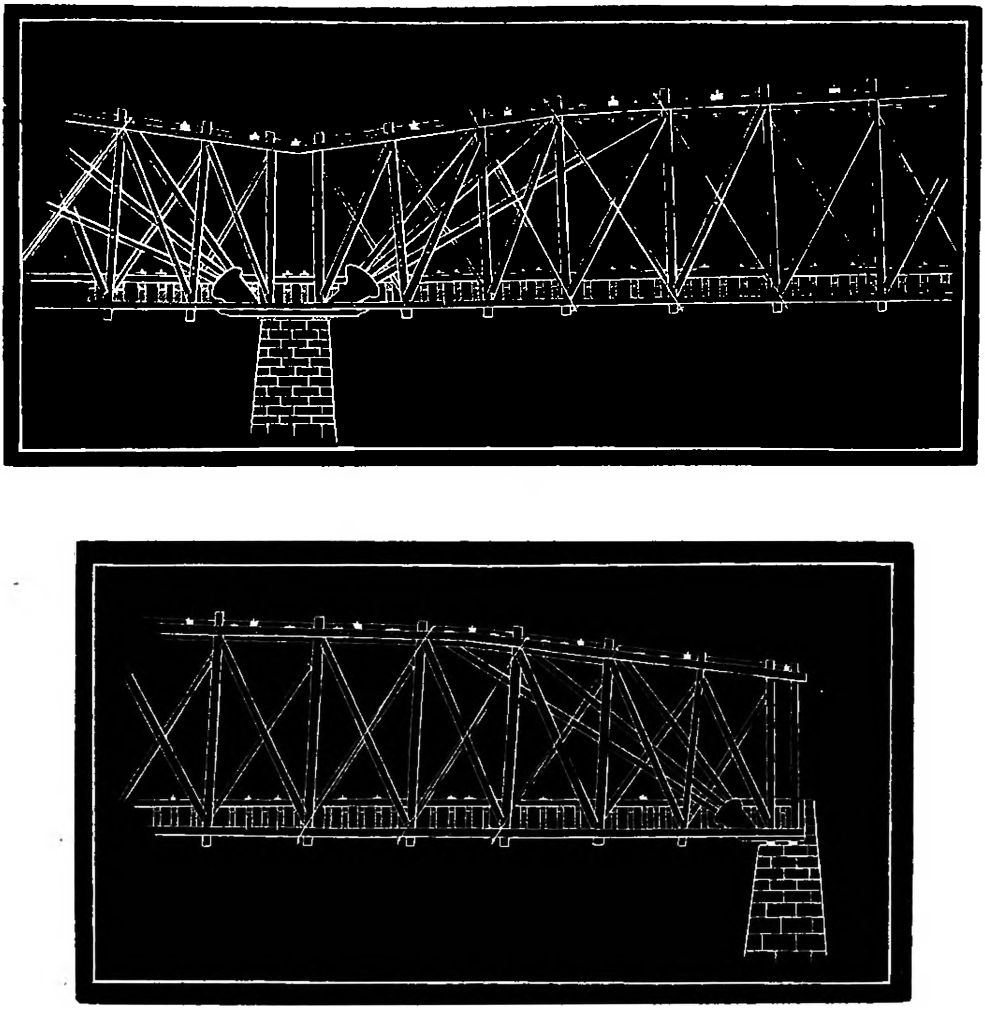
In this manner, a perfect equilibrium of forces is effected, as it is evident that the point G cannot change position, unless the points B B are thrust *outward* towards the ends of the truss, which must raise these points, this being prevented by the strain upon the points F F, communicated by the weight W, through the braces E E.

For a full plan of McCallum's Inflexible Arched Truss, the reader is referred to Fig. 97. Upon inspection, it will be observed that the sustaining principle is very much increased toward the ends of the truss, not only by the addition to the amount of material at these points, but it will be seen also that the panels become shorter as the vertical strain increases. The posts are placed upon lines radiating with the arch; the braces form equal angles with the posts; and in this way the latter are made to approach more nearly together toward the ends of the truss.

The student has already had sufficient evidence of the great strength of this form of truss, and it has also been shown that the tensile strain upon the lower chord is much less than in any other known plan. In fact, the latter may be *entirely severed*, and the structure will still be competent to sustain a heavy load. In this, it differs from all other combinations.

Upon referring to Fig. 97, which represents a clear span of 180 feet, it will be seen that the arch braces which rest upon the abutments are extended to points on the arch about forty-seven feet from the abutments. From the top of each set of arch braces, running diagonally on each side

Fig. 97.



of the truss, are placed heavy suspension rods, which are connected with the lower chords 12 feet further from the masonry. Thus the bridge seat is substantially transferred to a point 47 feet towards the centre of the bridge, reducing a span of 180 to 86 feet, so far as the *tensil strain* upon the lower chord is concerned.

For this intermediate space of 86 feet, the arch beam is of sufficient strength to sustain the whole load, if required.

Strength, however, is not all that is required, for a railroad bridge especially, subject as it is to a *moving* load;

there must also be *rigidity*, *stiffness*, freedom from vibration. A bridge *may* be strong yet flexible, rigid yet weak; in fact, flexibility is incompatible with durability; the structure should be prepared at all times to receive its load, and should not be permitted to change shape in the slightest degree by its passage over it.

To produce this result, an effective system of counter braces is indispensable.

The proper office of counter braces is frequently misunderstood, as is evident from the manner of their application in many cases in which they are used as *check braces only*, having a *negative* rather than a *positive* action; this may be readily shown. When the load is applied, the truss is deflected in consequence of the yielding of the braces; this has the effect of *shortening* the diagonals in the direction of their length, while the diagonals in the direction of the counter braces are correspondingly *lengthened*; this will leave a space between the ends of the latter, and the *bearing block* in the lower chord.

When the truss is in this condition, if wedges are inserted between the ends of the counter braces and the lower chord, in such a manner as to fill up the whole space, it is evident that the weight may be removed without at all affecting the shape of the truss, the deflection originally produced by the weight being maintained by the counter braces, the strain upon the sustaining braces and other portions of the truss remaining precisely the same as when the weight was suspended.

Now suppose the original weight to have been 200 tons, it is evident that, as soon as it is removed, each counter brace will be subjected to an upward thrust, easily found from its position; the sum of all the thrusts making 200 tons.

Now let there be a smaller load applied, this load will not produce any additional strain upon any portion of the truss, nor will the deflection be increased in the slightest degree; the only effect produced by suspending the latter weight will be the relief of the counter braces, equal to the difference between the first and second weights.

The inventor has found it very difficult to explain this clearly in the course of conversation with some individuals, from the fact that *weight* and *strain* were confounded. Now it is true, when the original weight was applied of 200 tons, the abutments were loaded with just 200 tons more than previously, and the truss was also loaded with ten tons more; but when the wedges were driven, and the weight removed, while the abutments were relieved of 200 tons pressure, the truss still retained the original strain produced, the weight being required to *produce the strain*, the latter remaining after the former has been removed.

In order to make a practical application of the above, the following method of adjusting the *Inflexible Arched Truss* is submitted. When these bridges are raised, it is usual to load them with a train of locomotive engines, attached closely to each other, and that greater weight may be obtained, the tenders are sometimes detached, and the bridge covered with engines only; with this load, the latter is strained down to a perfect bearing in all its parts; by this means the whole structure is more or less deflected, while the counter braces are hanging loosely in their places; if, therefore, when the bridge is in this condition with its load, the counter braces could be lengthened with considerable force, it would not recover its original shape upon removal of the load, but would be held down by the action of the counter braces to very nearly the same position as when loaded. In this plan of bridge, the lower ends of the counter braces rest in *iron stirrups*, which are attached to the vertical ties or posts at a point near the lower chord by means of castings and nuts, by which they may be lengthened several inches; in this manner they are made to perform a *positive* duty. When the bridge is adjusted as above, it is clear that a *less* load than that originally applied cannot produce any deflection whatever; the only effect of the passage of a train over it will be to relieve the counter braces, and will not add a pound pressure upon any timber of the trusses.

In the arrangement of any bridge truss the attainment of the following requisites is desirable:

First. Such equilibrium of forces as will produce uniformity of action.

Second. Such quantity and distribution of material as will ensure *a large surplus* of sustaining principle, *thereby guarding the structure against accident.*

Third. Perfect rigidity, that the combination in all its parts may have permanency *equal to the durability of the material composing the same.*

Fourth. The arrangement of the parts should be such as to be free, if possible, from *the necessity of adjustment.*

The McCallum *Inflexible Arched Truss* meets all these requirements.

PROBLEM.

Let it be required to find the equal weights w, w, Fig. 98, kept in a state of rest by a single weight, W, which has caused the arc q C a to assume the chord p D b, the rigidity of the arc being neglected.

Let R=the radius of the arc q C a, and put $2c$=its length; then the chord $q\,a$ and versine C D are readily calculated.

Let the chord $q\,a=2\,d$, and $\mathrm{C\,D}=2\,e$.

Let $f=\mathrm{A}\,q=\mathrm{A}\,p=\mathrm{B}\,a=\mathrm{B}\,b$ and

$$g=\mathrm{B}\,n=\mathrm{B}\,m=\mathrm{A}\,s=\mathrm{A}\,r.$$

Also, let $\mathrm{A\,B}=2\,h$

$$\frac{2\,h-2\,d}{2}=h-d,$$

$$\frac{h-d}{f}=\cos.\ \theta,\text{ putting }\theta\text{ for the angle }a\,\mathrm{B}\,n=q\,\mathrm{A}\,s.$$

$$\frac{2\,h-2\,c}{2}=h-c,$$

$$2\,h-2\,c=\mathrm{A\,B}-p\,b.$$

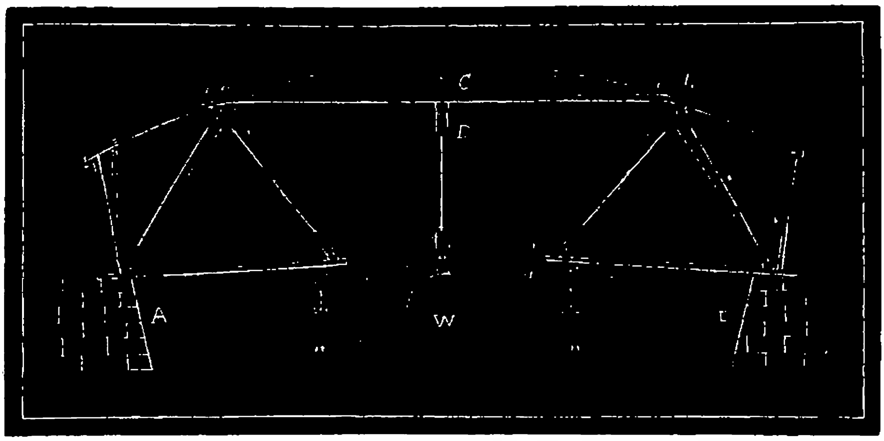
∴ Putting ϕ for the angle $b\,\mathrm{B}\,n=p\,\mathrm{A}\,s$,

$$\frac{h-c}{f}=\cos.\ \phi.$$

$$\therefore\quad \phi-\theta=\text{angle }a\,\mathrm{B}\,b=n\,\mathrm{B}\,m=s\,\mathrm{A}\,r.$$

$$\therefore\quad m\,n=g\sin.\ (\phi-\theta).$$

Fig. 98.



Then, according to the principle of work,

$$2\,g\,\text{sin.}\,(\phi-\theta)\,w=2\,e\,\mathbf{W}.$$

$$\therefore\quad w=\frac{e\,\mathrm{W}}{g\,\sin\,(\phi-\theta)}.$$

Let A B$=2\,h=188$ ft.; the length of the arc q C a=the straight line p D $b=2$ C$=136$ ft.; R$=200$ ft.; m B$=g=60$ ft; b B$=f=50$ ft.

It will be found that the arc q C a before being disturbed contains 38° 57′ 40″; the versine of this arc$=11{\cdot}449$ ft.$=2\,e$; the chord$=133{\cdot}39576$ ft.$=2\,d$.

$$\frac{h-d}{f}={\cdot}5460424=\text{cos. }56^\circ\,54'\,15''; \ (\theta).$$

$$\frac{h-c}{f}={\cdot}5200000=\text{cos. }58^\circ\,40'\,4''; \ (\phi).$$

$$\therefore\quad \phi-\theta=1^\circ\,45'\,49''.$$

$$\therefore\quad \frac{e}{g\,\text{sin.}\,(\phi-\theta)}=\frac{5{\cdot}7245}{60\,\text{sin.}\,(1^\circ\,45'\,49'')}=3{\cdot}100092.$$

$$\therefore\quad w=\text{more than 3 times W}.$$

Consequently, the weight W before reducing the arc q C a to the straight line p D b it must raise more than six times its own weight if posited at s and n.

DESCRIPTION OF AN IRON BRIDGE, IN WHICH THE FORCES ARE WELL COMBINED TO MEET THE DEMANDS OF RAILWAY TRAFFIC.

The bridge we now propose to describe is one belonging to a system of bridge-building introduced by Wendel Bollman; it was erected at Harper's Ferry, U.S., the practical working of which was carefully observed by the author of the present work. This iron suspension trussed bridge was 124 ft. between the abutments. The length of the cast iron in the stretcher was 128 ft. The weight of the cast iron, 65,137 lbs.; weight of wrought iron, 33,527 lbs.; making the total weight of cast and wrought iron, 98,664 lbs.

Fig. 99 is an elevation of part of the side, showing one pier and part of the cast-iron stretcher. The cap is removed from the pier to show how the rods are secured.

Fig. 99.

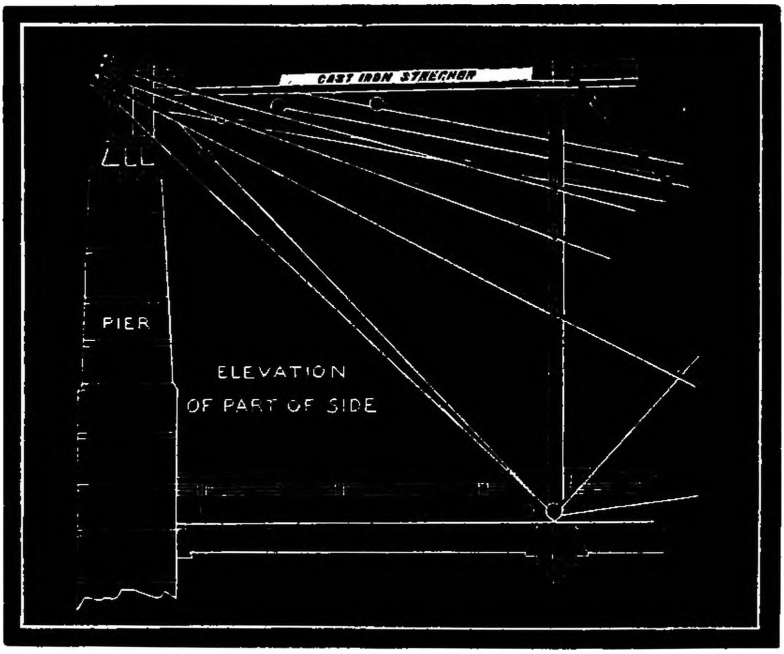


Fig. 100 is an elevation of a pier and four panels out of eight of which the bridge is composed. The system of arranging the braces and connecting rods is exhibited in this figure.

Fig. 100.

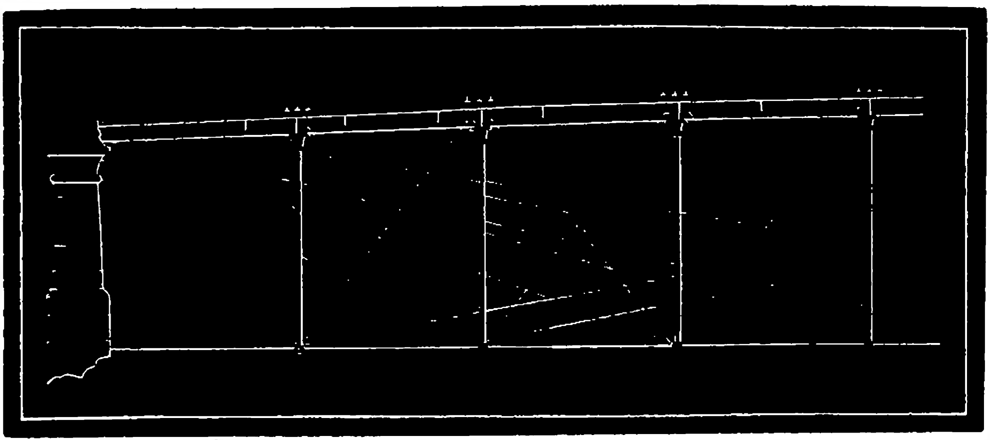


Fig. 101 is cross section, showing the floor bracing and the position of the rails. Fig. 101 also shows, in section, the roof and posts.

Fig. 101.

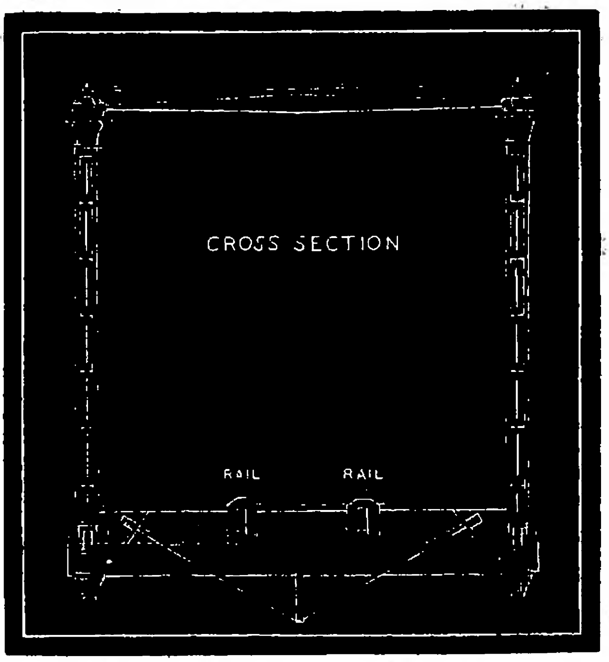


Fig. 102 shows a plan of the flooring of the bridge, the positions of the rails and floor bracing.

Fig. 102.

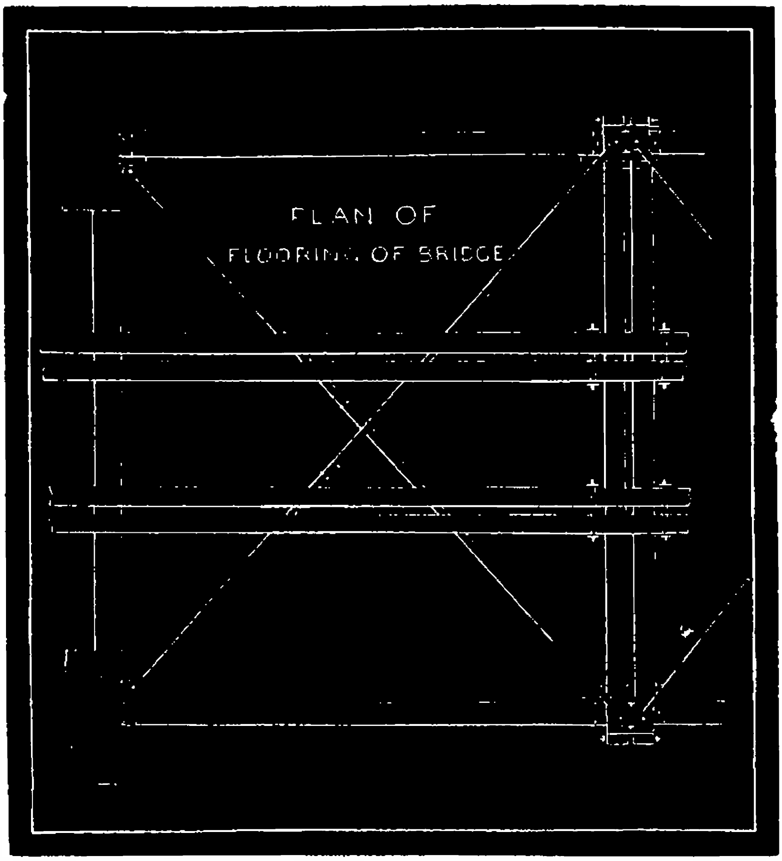
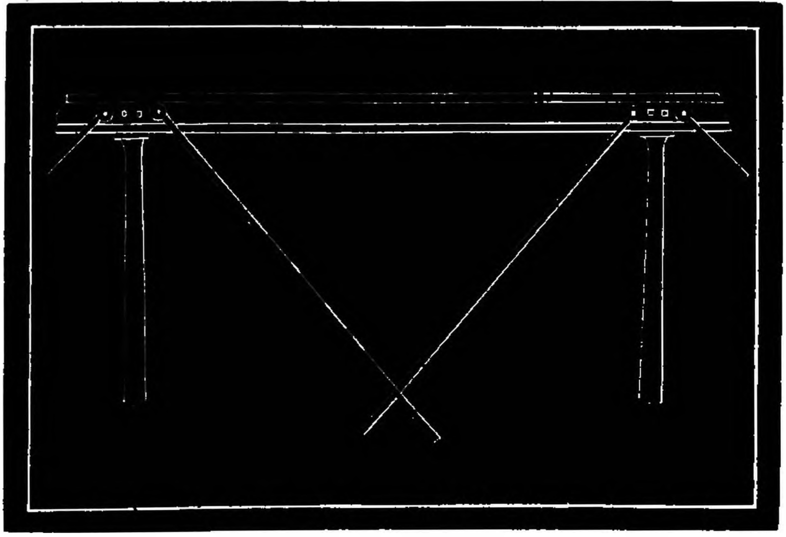

PLAN OF
FLOORING OF BRIDGE

Fig. 103 shows two posts, part of the stretcher, and the diagonal rods in one of the panels.

Fig. 103.



The wrought iron requires little workmanship, the rods from the centre to abutments having but an eye at one and a screw at the other end; with a weld or two between, according to length. The long counter rods have two knuckles and one swivel for adjustment of strain, and convenience in welding, as well as in raising the whole.

The cast-iron stretcher is octagonal without, circular within, and averages one inch of metal. It is cast in lengths according to the length of panel, and jointed in the simplest manner;—at one end of each length is a tenon, at the other a socket. The latter is bored out, and the tenon and its shoulder turned off in a lathe to fit the socket; thus, when thoroughly joined, to form one continuous pipe between abutments. The ends of the sections of cylinders, inserted to those contiguous, are slightly rounded, to allow a small angular movement without risk of joint fracture.

A cast-iron plate or washer sets on a bracket cast with each abutment end of stretcher, and at right angles to the centre acting rods. The tension bars are passed through this washer to receive a screw nut for the erection and adjustment of the system.

The stretcher or straining beam, the vertical posts, and suspension bars compose the essential features of the bridge; each post being hung by two bars from both ends of the stretcher independently of all the others; and each post and pair of tension bars forming with the stretcher a separate truss.

This system, perfect in itself, is additionally connected by diagonal rods in each panel; also by light hollow castings, acting as struts. The diagonal side rods might be safely dispensed with; for the peculiar merit of the truss is its perfect independence of such provision. They are therefore used as a safeguard only in case of the fracture of any of the principal suspension rods.

By this combination of cast and wrought iron, the former is in a state of compression, the latter in that of tension; the proper condition of the two metals. It unites the principles of the Suspension and of the Truss Bridges.

Each bar performs its own part in supporting the load in proportion to its distance from the abutment; so that the entire series of suspending rods transmits the same tension to the points of support as would be equally transmitted from thence to the centre of the bridge.

This bridge, it will be seen, is composed of seven independent trusses, which transfer the weight concentrated on each floor-beam directly to the abutments, without aid from any other connexion; and not from panel to panel as in general use.

The strain on cast and wrought iron is wholly in direct line; and the result, the least quantity of metal is required to carry a given weight. The weight of bridge and load has a vertical pressure on the piers, towers, &c., the only horizontal thrust being from the expansion of iron, which is accommodated by rollers, sliding on abutment bracket over its pedestal, or by other means: the necessary dimensions of masonry may therefore be most moderate.

It is evident, from an inspection of the cuts, that no chord is requisite at the bottom of the truss to resist tension; the only advantage of that employed is to regulate the movement produced by expansion, in the performance of which agency the resistance is one to compression.

Although the abutment bracket casting and its pedestal were so constructed as to admit of accommodation to expansion, by rollers, yet such contrivance was omitted with the view of fully testing the effect of greatest expansion throughout the system.

This bridge was inspected by the writer, ten months after it was erected at Harper's Ferry; during which time it had been exposed to extremes of cold and heat, and to an average run of twenty trains daily.

From the closest inspection, we find that the extreme expansion measures, as near as possible, five-sixteenths of an inch on each tower, or five-eighths in the entire length, 128 feet of stretcher; and without the slightest perceptible derangement of masonry; the dimensions of which are 4 feet square of base, 12 feet high, and 2 feet 9 inches at top.

While on the subject of expansion, it may be well to notice the effect from difference in expansion of the rods. At the first point of suspension, or where the longest and shortest rods meet, the counter rod is about four and a half times longer than the acting rod; and the expansion of the counter is four and a half times that of the acting rod. But there is also a proportionate difference in the lengths of stretcher from the point directly over the centre of connexion to the extremities of these rods. This has been practically proved in this bridge.

The suspender bolt, when the expansion is extreme or five-eighths of an inch in length of stretcher, exhibits a motive difference of three-sixteenths toward the short or acting rod; which difference is provided for, as seen by slot-dotted in Elevation, where the vertical suspender bolt moves to accommodate any such difference, and to give that proportion of weight to each rod according to the angle.

It affords easy access for repairs; for instance, should a new floor beam be required, it is but needed to slacken the horizontal rod and the keys in longitudinal strut, remove the washer under point of suspension, and let down the beam to be replaced: which can be done without trestling up any part of the bridge.

In case of fire, the floor may be entirely consumed without any injury to the side truss.

The permanent principle in bridge building, sustained throughout this mode of structure, and in which there is such gain in competition with every other, namely, the direct transfer of weight to the abutments, renders the calculation simple, the expense certain, and facilitates the erection of secure, economical, and durable structures.

Trial made on the 1*st day of June*, 1852, *to prove the capability of this Bridge.*

Three first-class tonnage engines, with three tenders, were first carefully weighed, and then run upon the bridge, at the same time nearly covering its whole length, and weighing in the aggregate 273,550 lbs., or $136\frac{1550}{2000}$ tons nett, being over a ton for each foot in length of the bridge.

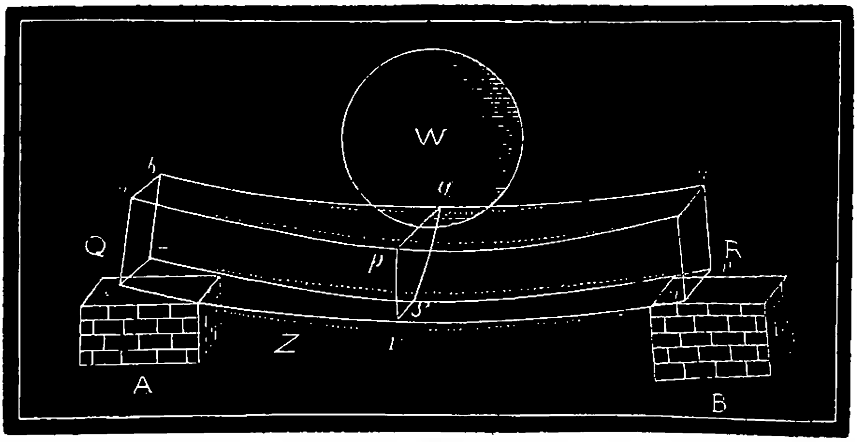
This burden was tried at about eight miles per hour, and the deflections, according to gauges properly set and reliable in their action, were at centre post, $1\frac{3}{8}''$, and at the first post from abutment nine-sixteenths of an inch.

From this test it is found that the load did not cover the entire length of bridge by about 13 feet, yet the excess of weight in the middle, and at a speed of about eight miles per hour, produced no greater deflection than $1\frac{3}{8}$ of an inch at the centre post, and nine-sixteenths of an inch at the first point from abutment.

Before proceeding further, it is necessary to point out some serious mistakes made by experimenters and writers on the strength of materials. When discussing the strength of girders resting on supports, the author of the present work, in his new theory of the strength of materials, first pointed out the fallacies involved through introducing an imaginary line, termed the neutral axis, and merely investigating upright laminæ of the material. We do not propose to discuss this subject thoroughly here, but to show the student how errors may be involved when the strength of girders is considered with respect to forces supposed to act only in parallel upright planes.

If a beam, Q R, Fig. 104, rests loosely on two supports, A and B, and is loaded in the middle with a weight, W, which deflects it; before the weight is placed on the beam $a\,b=p\,q=c\,d$; and $e\,f=r\,s=n\,m$; but when the beam is

Fig. 104.



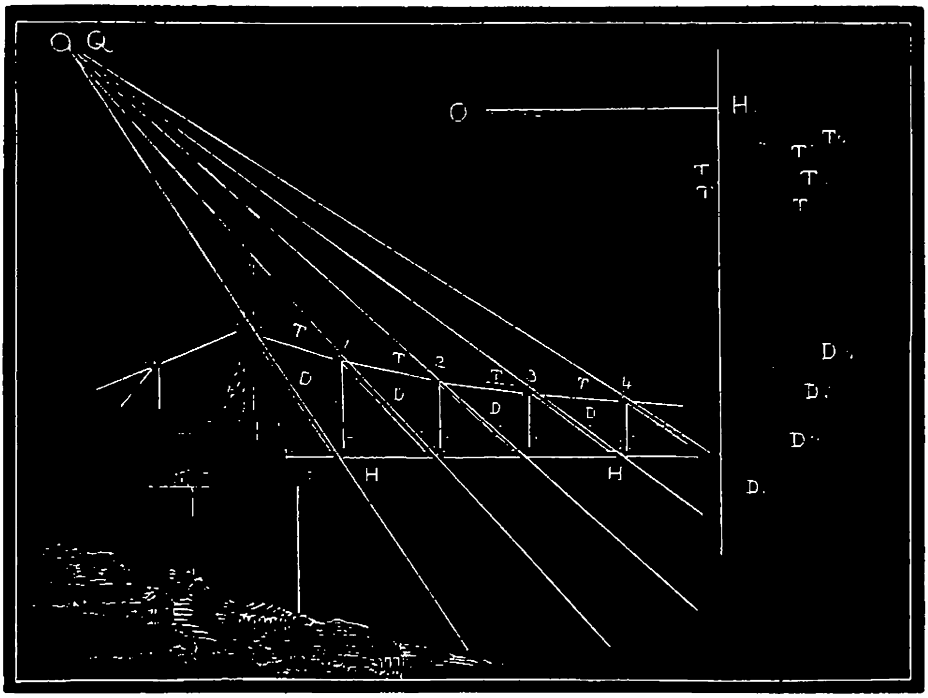
deflected by W, *p q* is greater than *a b* or *c d*, and *r s* is less than either *e f* or *m n*. Before the beam is loaded it is supposed to be rectangular; in most cases this change of form may be detected by experiment. Although the nature of the material and amount of pressure may render this change of form imperceptible, yet these forces acting across the girder, in the directions of *p q*, *r s*, are in operation, loosening bolts, buckling and puckering upright sheets, and so on. This action should be carefully attended to by engineers in constructing girders, whether solid, hollow, or composed of skeleton frames.

The material at *r s* is wire-drawn and compressed, while at *p q* the material becomes upset, extended, and loosened, according to the elastic limit and nature of the girder. The current erroneous theory of the strength of materials supposes, when the beam is bent by a weight, W, the fibres are compressed at *p q* and extended at *r s*, without alteration of breadth; that is *p q* remains$=$*a b* or *c d*, and also$=$*r s* or *e f*. When Robert Stephenson was erecting the Menai tubular bridge in Wales, the author of the present work, Oliver Byrne, published an outline of his new theory of the strength of materials in the *Civil Engineer and Architects' Journal*, and explained his views to Mr. Stephenson, who directed the author to observe his (Stephenson's) final experiment on a model of the bridge, one-sixth the full size. Fairbairn and other engineers were present. The direction of the fracture depends on the molecular action of the material of which the girder is formed. A portion of the body will often be forced out near the line *p q*; but when the substance supporting the weight is tough, the separation may take place irregularly and diagonally, with a sliding cutting motion, and not directly through the plane *p q r s*, in the middle.

METHOD OF CONSTRUCTING SKELETON STRUCTURES, PROPOSED AND PATENTED BY EDWARD KOCH.

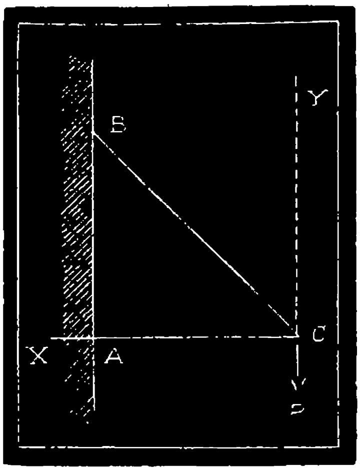
Half the suspended beam, Fig. 105, gives a clear representation of most of the particulars of this mode of construction, the practical application of which depends on the connecting bolts filling their respective holes without play. After these bolts are posited with mathematical accuracy, it is also required that neither bolts nor holes afterwards wear much. The inventor proposes plans to surmount these and other engineering difficulties, in a book by his friend Olaus Henrici, entitled "Skeleton Structures, especially in their Application to the Building of Steel and Iron Bridges."

Fig. 105.



In the diagram of forces, Fig. 105, O H_1 is parallel and H_1 D_1 perpendicular to H H; the diagonal braces, D, D, D, meet in a point Q; T_1, T_2, T_3 . . . are parallel to T, T, T, . . . respectively; and D_1, D_2, D_3, . . . are drawn parallel to D, D, D, . . . respectively. The construction of this diagram will show how the forces act in magnitude and direction, and may be measured on a scale of equal parts, or found by calculation, when the inclinations of the bars are given. But, as the triangles standing on H_1 H_1 are right-angled triangles, Mr. Henrici employs the following method of estimating the comparative forces:

Fig. 106.



In the structure, Fig. 106, A C and B C represent bars which are connected at A and B by bolts to a wall piece, and at C by bolt to each other. A weight P acts on the bolt C, and we want to ascertain the strain in the two bars. Firstly, we have to apply two rectangular axes through C, and because A C is horizontal, and C P vertical, we take these directions as the axes. The strain in the bar B C we call S, and the one in A C, T. If S is resolved according to the axes, into its components, we find

$$\text{in the direction of } CX:\ S\frac{AC}{BC},$$

$$\text{,,} \quad \text{,,} \quad CY:\ S\frac{AB}{BC}.$$

The two other forces act in the direction of the axes; hence we get

$$\text{in the direction of } CX:\ S\frac{AB}{BC}+T,$$

$$\text{,,} \quad \text{,,} \quad CY:\ S\frac{AB}{BC}-P,$$

where P is subtracted, because it acts contrary to the strain S. Both components must disappear; we have therefore:

$$S\frac{AC}{BC}+T=0,$$

$$S\frac{AB}{BC}-P=0.$$

From these equations we see that

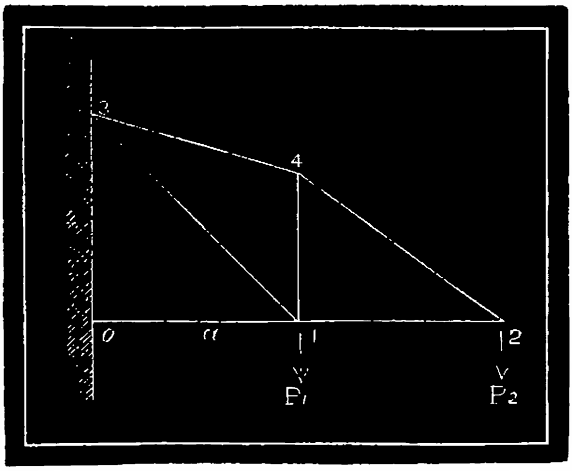
$$S=\frac{BC}{AB}\cdot P,$$

$$T=-\frac{AC}{AB}\cdot P;$$

which shows, that in the bar B C there is tension and in A C compression, because T is negative.

In this example we had only one connexion-point; we shall take, therefore, a more complicated example by joining more bars to those, and thus we get a structure like the one presented in Fig. 107, of which we shall proceed

Fig. 107.



to calculate the strains. We will call here the connexion-points 0, 1, 2, 3, &c., and the bars will be represented by two figures (01), (12), &c.; the lengths of the bars we call r with the respective figures, r_{13}, &c., and the strains likewise, S_{13}, &c. We take now

$$r_{01}=r_{12}=a,$$

$$r_{03}=a,$$

$$r_{14}=\frac{3}{4}a,$$

and find further:

$$(r_{13})^2=a^2+a^2=2a^2,$$

$$(r_{34})^2=a^2+\left(\frac{a}{4}\right)^2=a^2\left(1+\frac{1}{16}\right)$$

$$= \frac{17}{16} a^2,$$

$$(r_{24})^2 = a^2 + \left(\frac{3a}{4}\right)^2 = a^2\left(1 + \frac{9}{16}\right)$$

$$= \frac{25}{16} a^2;$$

therefore:

$$r_{13} = a\sqrt{2},$$

$$r_{34} = a\frac{\sqrt{17}}{4},$$

$$r_{24} = \frac{5}{4} a.$$

By this all bars are determined in length and position, and we can proceed to resolve the forces. For axes we always take one horizontal and one vertical line, and because we have three free connexion-points, 1, 2, and 4, we can form six equations, by which we are able to ascertain all the six strains at once. For point 2 we have the same equations as in the previous example, by introducing S_{24} instead of S; and S_{12} instead of T; as likewise r_{14}, r_{12}, and r_{24} instead of A B, A C, and B C; and we get

$$S_{24} = \frac{r_{24}}{r_{14}} . P_2,$$

$$S_{12} = -\frac{r_{12}}{r_{14}} . P_2;$$

or by introducing the values of r:

$$S_{24} = \frac{5}{3} P_2,$$

$$S_{12} = -\frac{4}{3} P_2.$$

Looking at point 4, we get:

S_{24} resolved into $S_{24} \cdot \frac{r_{12}}{r_{24}}$ and $S_{24} \cdot \frac{r_{14}}{r_{24}}$,

S_{34} resolved into $S_{34} \,.\, \frac{r_{01}}{r_{34}}$ and $S_{34} \,.\, \frac{r_{03}-r_{14}}{r_{34}}$

S_{14} ,, ,, 0 ,, S_{14}.

By adding these, we get 2 equations:

$$S_{24} \frac{r_{12}}{r_{24}} - S_{34} \frac{r_{01}}{r_{34}} = 0,$$

$$S_{34} \frac{r_{14}}{r_{24}} - S_{34} \frac{r_{03} - r_{24}}{r_{34}} + S_{14} = 0.$$

The first equation will give

$$S_{34} = \frac{r_{34}}{r_{01}} \,.\, \frac{r_{12}}{r_{24}} \,.\, S_{24},$$

or because

$$S_{24} = \frac{5}{3} P_2 \text{ and } \frac{r_{34}}{r_{01}} \,.\, \frac{r_{12}}{r_{24}} = \frac{\sqrt{17}}{5},$$

$$S_{34} = \frac{\sqrt{17}}{3} P_2.$$

From the other equation follows

$$S_{14} = -\frac{2}{3} P_2.$$

We have still to ascertain the strains S_{01} and S_{13}, which we are able to do by the equations of point (1), after having altered them in the same way:

$$S_{01} + \frac{r_{01}}{r_{13}} \,.\, S_{13} - S_{12} = 0,$$

$$S_{14} + \frac{r_{03}}{r_{13}} \,.\, S_{13} = P_1.$$

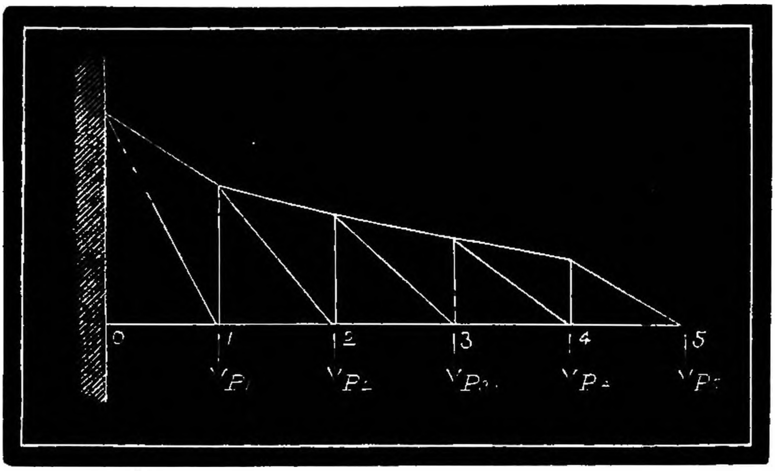
By which, in connexion with the above equations, we find;

$$+_{13} = \sqrt{2}\, P_1 - \frac{3}{4}\sqrt{2}\, P_2.$$

$$S_{01} = -P_1 - \frac{7}{12} P_2.$$

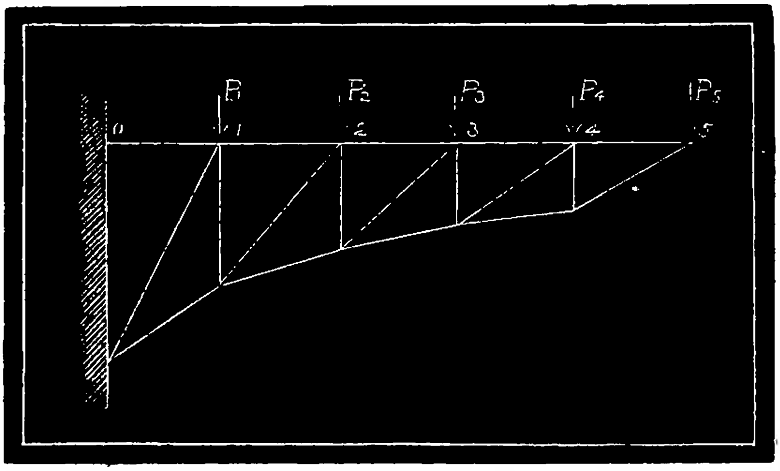
It will be clearly understood how these calculations are carried on. Taking a structure like that in Fig. 108, the calculations are done in the same way; one strain is calculated after the other without difficulty.

Fig. 108.



I will mention here at once, that the same calculation can be used for a trussed structure, as in Fig. 109; the structure is exactly the reverse of Fig. 108, and it is not necessary to alter the formulæ; for as all the Ps act here in the contrary direction, all the signs will be reversed. All the strains, therefore, will be the reverse of those in Fig. 108; where there was tension in a bar there is now compression; and where there was compression, tension.

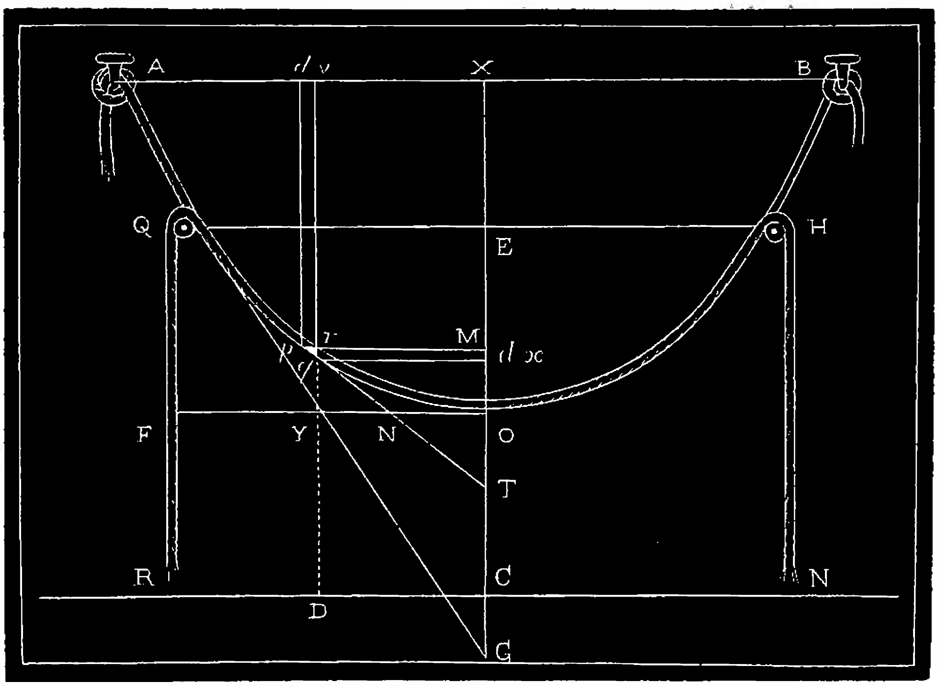
Fig. 109.



CHAIN AND SUSPENSION BRIDGES.

Suppose a perfectly flexible chain or cord of uniform density and thickness to be suspended from two fixed points, A and B, Fig. 110, and when in equilibrium to form the curve A O B; this curve is termed the *Catenary*. The equations of this curve, of so much importance in mechanics, being of a mixed exponential kind, mathematicians were obliged to resort to guessing and all sorts of dodges, to obtain results which were often very far from the truth. The results here referred to can now be obtained with the greatest ease by direct processes of the dual calculus, in an endless variety of ways, without the use of tables. Having thus pointedly arrested the attention of students in unmistakable terms, at the same time it may be necessary to inform them, that here, as elsewhere, we accommodate those inexperienced in such inquiries with solutions under simple forms easily intelligible.

Fig. 110.



PROPOSITION.

To find the Equations of the Catenary Curve.

Let O be the lowest point of the chain or cord A O B, Fig. 110, O $M = x$; M $r = y$; and the length of the arc O $r = s$. Again, let v be the length of a portion of the chain, the weight of which is equal to the tension at O. If we suppose the part O r rigid, after it has assumed the form of equilibrium, it will evidently be supported in the same manner, and the tensions at O and $p\,q\,r$ will be the same as when it swung loosely suspended from the points A, B. O $p\,q\,r$ is therefore kept at rest by three forces, namely, the tension at O acting in the direction of the tangent O Y, the tension at the point $p\,q\,r$, acting in the direction of $p\,q$ T, the tangent to the curve at the point $p\,q\,r$, and the weight of the piece of cord or chain O r, acting in a vertical direction. Because the three forces just described are respectively in the directions to the three sides of the triangle T p M, the forces being in equilibrium will be proportional to these sides.

Hence, (Weight of O r) : (Tension at O) : : T M : M p. $p\,q\,r$ represent a very small right-angled triangle, similar to the larger right-angled triangle T M p; in the language of the differential calculus $q\,r$ is represented by $d\,x$, $r\,p$ by $d\,y$, and $p\,q$ by $d\,s$. Whence, v being put for a piece of cord or chain, the weight of which is equal the tension at O in the direction of the tangent O Y; therefore,

$$s : v :: dx : dy$$

$$\therefore \quad \frac{dy}{dx} = \frac{v}{s}.$$

In every plane curve $(d\,s)^2 = (d\,x)^2 + (d\,y)^2$, and, consequently,

$$\frac{ds}{dx} = \left(1 + \frac{dy^2}{dx^2}\right)^{\frac{1}{2}}.$$

$$\therefore \quad \text{In the } \textit{catenary} \ \frac{ds}{dx} = \frac{\sqrt{v^2 + s^2}}{s}, \text{ or } dx = \frac{s\,ds}{\sqrt{v^2 + s^2}}.$$

Taking the integral of this last equation, and observing that $s=o$ when $x=o$, we obtain

$$x+v=\sqrt{v^2+s^2}, \quad \text{or} \quad s^2=x^2+2\,v\,x; \qquad \text{(I.)}$$

When v is determined, (I.) is the equation of the *catenary* expressed by x and s as variables.

Again, because $\frac{d\,y}{d\,x}=\frac{v}{s}=\frac{v}{\sqrt{x^2+2\,v\,x}}$, which, being integrated, gives $\frac{y}{v}=\log_{\epsilon}.\ \frac{x+v+\sqrt{x^2+2\,v\,x}}{v}$

$$\therefore\ \epsilon^{\frac{y}{v}}=\frac{x+v}{v}+\frac{\sqrt{x^2+2\,v\,x}}{v}; \qquad \text{(II.)}$$

ϵ being put for the base of the hyperbolic system of logarithms, (II.) is an equation to the curve between the variable co-ordinates x and y; v being unknown, but constant. The dual logarithm of $\epsilon = 10^8$, written $\downarrow,(\epsilon)=100000000$,

$$\epsilon=2{\cdot}718281828\ldots.$$

Transposing $\frac{x+v}{v}$, and squaring both sides of (II.), we obtain

$$\epsilon^{\frac{2y}{v}}-2\,\epsilon^{\frac{y}{v}}\left(\frac{x+v}{v}\right)+\frac{(x+v)^2}{v^2}=\frac{x^2+2\,v\,x}{v^2}$$

$$\therefore\ x+v=\frac{v}{2}\left(\epsilon^{\frac{y}{v}}+\frac{1}{\epsilon^{\frac{y}{v}}}\right); \qquad \text{(III.)}$$

Further, because, (I.) $s^2=(x+v)^2-v^2$

$$\therefore\ \left(\epsilon^{\frac{y}{v}}-\frac{x+v}{2}\right)^2=\frac{s^2}{v}$$

$$\therefore\ s=\frac{v}{2}\left(\epsilon^{\frac{y}{v}}-\epsilon^{-\frac{y}{v}}\right); \qquad \text{(IV.)}$$

(IV.) is an equation to the *catenary* between the variables s and y; it is to be particularly observed that v is unknown in (I.), (II.), (III.), (IV.), but not variable. Suppose t to be the length of a portion of the cord or chain which is equal to the tension of Q, then $OE = x$, $EQ = y$, $OQ = s$, and

$$t : s :: QG : GE :: ds : dx$$

$$\therefore \quad s\,ds = t\,dx.$$

Differentiating (I.), or $s^2 = x^2 + 2\,v\,x$, gives $s\,d\,s = x\,d\,x + v\,d\,x$

$$\therefore \quad t\,dx = x\,d\,x + v\,d\,x;$$ dividing by $d\,x$ gives

$$t = x + v.$$

Now, suppose the tension at Q to be balanced by means of Q R, a portion of the chain passing over a pulley at Q and hanging freely; then $QR = x + v = OE + v$; consequently, $FR = YD = OC = v$, evidently a constant quantity, although unknown. Hence, if the tension be supposed to be balanced by means of portions of the chain or flexible body hanging over pulleys at the points $p\,q\,r$, Q, A, the lower ends will be in the same horizontal line, R, D, C.

Ques. 63. A heavy flexible chain, each foot in length weighing 20 lbs., is suspended at its extremities to two fixed points, in the same horizontal line, 106·2 feet asunder, the greatest depth of the curve being 15·8 feet. It is required to find the length of the curve, the tension at the lowest part, and the tensions at the points of suspension, by direct calculation, without the use of tables, or methods of approximation?

In dual arithmetic we have three corresponding numbers, namely, natural number, dual number, and dual logarithm; any one of these corresponding numbers being given, the other two may be found by a few simple additions and subtractions.—See the "Young Dual Arithmetician."

We have first to put equation (III.) in form to be operated upon by dual arithmetic.

$$x + v = \frac{v}{2}\left\{ \epsilon^{\frac{y}{v}} + \frac{1}{\epsilon^{\frac{y}{v}}} \right\}; \quad \text{(III.)}$$

When y is put $=1$, x may be represented c^2, x and y being both given to find the value of v. Hence (III.) becomes

$$2c^2\left(\frac{1}{v}\right)+2=\epsilon^{\frac{1}{v}}+\frac{1}{\epsilon^{\frac{1}{v}}}$$

$$\therefore\quad 2c^2\left(\frac{1}{v}\right)=\epsilon^{\frac{1}{v}}-2+\frac{1}{\epsilon^{\frac{1}{2v}}}=\left\{\epsilon^{\frac{1}{2v}}-\frac{1}{\epsilon^{\frac{1}{2v}}}\right\}^2$$

$$\therefore\quad c\sqrt{2}\left(\frac{1}{v}\right)^{\frac{1}{2}}=\epsilon^{\frac{1}{2v}}-\frac{1}{\epsilon^{\frac{1}{2v}}};\text{ or,}$$

$$2c\left(\frac{1}{2v}\right)^{\frac{1}{2}}=\epsilon^{\frac{1}{2v}}-\frac{1}{\epsilon^{\frac{1}{2v}}}.$$

Putting $\frac{1}{2v}=z$, the last becomes

$$2c\sqrt{z}=\epsilon^{z}-\epsilon^{\frac{1}{z}},\qquad\text{(V.)}$$

Question the 63rd being the first of the kind solved by an independent and direct process, without employing tables or approximate methods of trial and error, it is therefore necessary that we should be particular with respect to details.

The dual logarithm of 2 is the whole number 69314718, written,

$$\downarrow,\ (2)=69314718,\ .$$

This logarithm is of the ascending branch, marked by a comma on the right of the logarithm below, and immediately after the arrow. The dual logarithm of the reciprocal of 2· or of ½ is represented by the same whole number '69314718, and written,

$$\downarrow,\ (\tfrac{1}{2})={}'69314718.$$

This logarithm is of the descending branch, marked by a comma to left of the logarithm above. When 69314718, is considered positive, '69314718 is considered negative, and *vice versâ.*

Natural Numbers.	Dual Numbers.	Dual Logarithms.
2·00000000	↓ 7,2,6,0,7,8,2,6,	69314718,
·50000000	↑ '6'6'0'6'8'2'0'2	'69314718

$$\downarrow,(10) = 230258509, \quad \text{and} \quad \downarrow,\left(\frac{1}{10}\right) = {}'230258509.$$

The same notation and arrangement is applied to other natural numbers and their reciprocals. See the "Young Dual Arithmetician," and "Dual Arithmetic: a New Art," Parts I. and II.

In the question before us, $c^2 = \dfrac{15\cdot8}{53\cdot1} = \cdot29755178907$;

$\therefore \quad \downarrow,(2\,c) = \downarrow,(1\cdot09096616) = \downarrow,0,8,7,4,6,4,5,5, = 8706369,$ and (V.) becomes

$$\epsilon^{z} - \epsilon^{\frac{1}{z}} = 1\cdot09096616\sqrt{z}$$

$$\downarrow,(\epsilon) = 10^8 = 100000000,$$

Then, putting u_1, for the first dual digit of $z\downarrow_{,}(\epsilon)_{,}$ and $'u_1$ for the first dual digit of the logarithm of $-z\downarrow_{,}(\epsilon)$ the reciprocal to $z\downarrow_{,}(\epsilon)$; $'u_1$ is nearly $= u_1$, but less than u_1.

To place this matter in a clear light, it is only necessary to observe that,

$$\text{If } x = \downarrow 1,0,0,0,0,0,0,0,\text{then}\frac{1}{x} = \downarrow{}'1\ 1,0,1,0,0,0,1,;\ (9531018).$$

$$,,\ x^2 = \downarrow\ 2,0,0,0,0,0,0,0,\ ,,\ \frac{1}{x^2} = \downarrow{}'2\ 2,0,2,0,0,0,2,;\ (19062036).$$

$$,,\ x^3 = \downarrow\ 3,0,0,0,0,0,0,0,\ ,,\ \frac{1}{x^3} = \downarrow{}'3\ 3,0,3,0,0,0,3,;\ (28593054).$$

&c. &c.

$$\downarrow,(x) = \downarrow,(1\cdot1) = \downarrow,1,0,0,0,0,0,0,0, = 9531018,\ ;$$

$$\downarrow,(\frac{1}{x}) = \downarrow,(\frac{1}{1\cdot1}) = \downarrow,{}'1\ 1,0,1,0,0,0,1, = {}'9531018\ ;\ \&c.$$

These things being premised, we have

$$1\ \downarrow[u_1 - 1\ \uparrow[u'_1 = 2c\sqrt{z} = 2c\sqrt{u_1 \times (\cdot09531018)};$$

but since u_1 is nearly equal to u'_1, in order to find a convenient value for u_1 the last equation may be put under the form,

$$+2\left(\frac{u_1}{10}\right) = 2\,c\sqrt{u_1 \times (\cdot09531018)}$$

$$\therefore\quad \frac{{u_1}^2}{100} = c^2 u_1\,(\cdot09531018);\ \text{or } u_1 = 100c^2\,(\cdot09531018);$$

which gives $u_1 = (\cdot29755\ldots)^{\cdot}(100)\ (\cdot0953\ldots) = 2\cdot835\ldots\ldots$ Because $2\cdot835\ldots$ is less than a convenient value for u_1, but greater than the corresponding value of u'_1; consequently $\downarrow 3$, is a convenient value for u_1. By pursuing a similar process of reasoning, z as far as the fourth dual digit will be found to be equal $\downarrow 3,0,3,4,u_5,;$ the next step gives the four succeeding digits, and at the same time develops the general plan of operating on such equations; besides, any mistake that may be made in the preceding operations will be detected.

$$\downarrow,(1\cdot33553107) = \downarrow,3,0,3,4,0,0,0,0, = 28932904,$$

$$\text{Reciprocal, }{}'28932904 = \uparrow{}'2'7'8'2'5'1'6'5 = \downarrow,(\cdot74876579).$$

Putting z_1 for the dual logarithm of $\downarrow 3,0,3,4, u_5, \ldots$ divided by 10^8, or

$$z_1 = \cdot 28932904 + \cdot 00001000\, u_5 ;$$

now equation (V.) becomes

$$\epsilon^{z_1} - \frac{1}{\epsilon^{z_1}} = 2c \sqrt{\cdot 28932904 + \cdot 00001000\, u_5} =$$

$$2c \sqrt{\cdot 28932904} \sqrt{1 + \cdot 00003803\, u_5} =$$

$$(\downarrow 0,8,7,4,6,4,5,5,) \sqrt{\cdot 28932904}\, (1 + \cdot 00001902\, u_5) =$$

$$\cdot 58682320\, (1 + \cdot 00001902\, u_5)$$

$$\begin{array}{r|l} \underline{\cdot 58682} & \underline{320} \\ & 587 \\ & 528 \\ & 1 \end{array} \qquad \left\{ \begin{array}{l} \text{because } 1 + \cdot 00001902 \text{ may} \\ \text{be taken for the square} \\ \text{root of } 1 + \cdot 00003803. \end{array} \right\}$$

$$\cdot 00001\overline{116} \text{ for } \cdot 0001902$$

$$\therefore\ 2c \sqrt{\cdot 28932904 + \cdot 00001000\, u_5} = \cdot 58682320 + \cdot 00001116\, u_5$$

$$\epsilon^{z_1} - \frac{1}{\epsilon^{z_1}} = 1 \cdot 33553107\, (1 \overset{+}{\downarrow} [u_5) - \cdot 74876579\, (1 \underset{+}{\uparrow} [u'_5) ;$$

$$1 \cdot 33553107\, (1 \overset{+}{\downarrow} [u_5) \text{ may be put} =$$

$$1 \cdot 33553107 (1 + \cdot 00001000\, u_5),$$

$$\text{and} - \cdot 74876579\, (1 \underset{+}{\uparrow} [u^1_5) \text{ may be put} =$$

$$- \cdot 74876579 (1 + \cdot 00001000\, u_5) ;$$

$$\therefore\ \epsilon^{z_1} - \frac{1}{\epsilon^{z_1}} = \cdot 58675628 + 2 \cdot 08429686\, (\cdot 00001000\, u_5)$$

$$\begin{array}{r|l} 2 \cdot 0842 & \underline{9686} \\ & 2084 \end{array}$$

$$\therefore\ \cdot 58675628 + \cdot 00002084\, u_5 = \cdot 50682320 + \cdot 000011\overline{1}6\, u_5$$

$$\therefore \quad u_5=\frac{5792}{968}=5{\cdot}983$$

$$10^8z=\downarrow, 3,0,3,4,5,9,8,3,=28938887,$$

$$\therefore \quad z=\frac{1}{2v}={\cdot}28938887$$

$\therefore$ $v=1{\cdot}72777897$ true to the last figure, when $y=1$; $\therefore$ when $y=53{\cdot}1$, $v=91{\cdot}745063$ ft.

From (I.), $s^2=x\,(x+2v)=15{\cdot}8\,(15{\cdot}8+183{\cdot}490126)$

$$\therefore \quad s=56{\cdot}11395$$

$\therefore$ the length of the whole curve, Q O H, Fig. 110, is 112·2279 ft.

$$x+v=107{\cdot}545 \text{ ft.}$$

Tension at the lowest part $=91{\cdot}745\times20=1834{\cdot}9$ lbs.

Tensions at points of suspension $=107{\cdot}545\times20=2150{\cdot}9$ lbs.

Ques. 65. The length of a heavy flexible chain A O B, Fig. 110, is just double the horizontal line A B, joining the points of suspension A and B; required the pressure on these points and the tension at the lowest point O; the distance of O from the horizontal line A B is also required?

If A X = 1, the length of half the curve, or A Q O, = 2, and equation (IV.), becomes

$$2=\frac{v}{2}\left(\epsilon^{\frac{1}{v}}-\frac{1}{\epsilon^{\frac{1}{v}}}\right); \text{ and putting } z=\frac{1}{v}$$

$$\epsilon^{z}-\frac{1}{\epsilon^{z}}=4z.$$

When $z=2{\cdot}$ the dual logarithm of ϵ^{z} is nearly = the dual log. of 8·; $4z=8{\cdot}$ also. It may be remarked here that, in operating with the dual calculus, we may take dual digits much greater or much less than any particular one pointed out, and yet obtain, without tentative artifices, a result as near the truth as we please. Since this method gives the same result as near the truth as we please, by several direct processes, it presents a series of direct operations, and not a succession of approximate trials.

$\downarrow$(7·38905604)=200000000, and $\downarrow$,(·13533528)='200000000; consequently we may assume

7·38905604 (1 ↧ [u_1)—·13533528(1 ↥ [u'_1)=
4(2·00000000+·09531018 u_1); (A)

200000000, is increased by 9531018, for every unit in u_1; and 2·00000000 by ·09531018.

To find a convenient value for u_1, (A) may become

$$7{\cdot}25372076+7{\cdot}52439132\,\frac{u_1}{10}=8{\cdot}00000000+{\cdot}38124072u_1$$

$$\therefore\quad {\cdot}37119841\,u_1={\cdot}74627924$$

$$\therefore\quad u_1=+2 \text{ or } \downarrow 2,$$

In practice, one tenth of the figures here employed would not be required to find that $\downarrow$ 2, is a convenient value for u_1. This may be too great, but that is of no consequence, as the next digit may be negative to compensate for the excess in taking u_1= $\downarrow$ 2,

200000000,+19062036,= $\downarrow$, (8·94075744) and '219062036= $\downarrow$, (·11184734).

To find a convenient value of u_2, the equation now takes the form,

8·96075744 (1 ↧ [u_2)–·11184734 (1 ↥ [u_2)
=4(2·19062136+·00995033 u_2);

therefore we may put

$$8{\cdot}82891010+9{\cdot}05260478\frac{u_2}{100}=8{\cdot}76248144+{\cdot}03980132\,u_2$$

$$\therefore\quad u_2,=-1{\cdot} \text{ or } = \downarrow 0,\ '1.$$

For every unit in u_2, we employ 995033, and ·00995033; but for every unit in $'u_2$ we employ '1005034 and $-{\cdot}01005034$. See "The Young Dual Arithemetician," pp. 82, 99.

$$\begin{array}{r} 219062036, \\ '1005034 \\ \hline 218057002, \end{array} = \downarrow, (8{\cdot}85134984);$$

$$'218057002 = \downarrow, ({\cdot}112977175)$$

$$\therefore \quad 8{\cdot}85134984\,(1 \downarrow [u_3) - {\cdot}11297718(1 \uparrow [u_3) =$$

$$4(2{\cdot}18057002 + {\cdot}00099950\, u_3)$$

$$\text{This gives } u_3 = -3 \text{ or } \downarrow '0\ '0\ '3$$

This process being continued gives $z = 2 \downarrow 2, '1\ '3\ '3\ 5, 0, 5, 2,$

the log of ϵ^z is 217731902, for $\downarrow, 2, '1'3'3\,5, 0, 5, 2, = 17731902$. Putting B for the base 1·00000001, then $B^{217731902} = \epsilon^{2.17731902}$.

The reciprocal of $2{\cdot}17731902 = {\cdot}45928042 = v$, the length of chain = tension at O.

But as $x^2 + 2\,v\,x = s^2$; $x = 1{\cdot}592779 = XO$.

$x + v = 2{\cdot}052059$ = the length of the chain that amounts to the tension at A and B.

Ques. 66. Given the length of a heavy flexible chain R Q O H N = 1130 ft. ($2a$), Fig. 110; this chain hangs freely over two pulleys Q, H in the same horizontal line Q H, = 226 ft. ($2b$); required the position in which it will rest, the length of Q R, and the length of chain that is equal to the tension at O?

$$\text{Let } QE : OQR :: 1 : n \text{ or } b : a :: 1 : n.$$

It is evident that when the chain is in a state of rest, what in the foregoing questions was the pressure on the points of suspension Q, H, will be equal to the weight of either H N or of $QR = x + v$, the parts hanging vertically. Putting s = the length of Q O, when $QE = 1 = y$; $OQR = n$; and $QF = EO = x$, then $x + v + s = n$; $v = FR = OC$.

$$\left.\begin{aligned}&\text{From (III.).}\quad x+v=\frac{v}{2}\left(\epsilon^{\frac{1}{v}}+\frac{1}{\epsilon^{\frac{1}{v}}}\right)\\&\text{From (IV.).}\quad s=\frac{v}{2}\left(\epsilon^{\frac{1}{v}}-\frac{1}{\epsilon^{\frac{1}{v}}}\right)\end{aligned}\right\}\text{, since } y=1.$$

Whence, from adding these equations together, we obtain

$$x+v+s=n=v\epsilon^{\frac{1}{v}}.$$

$$\therefore\quad \downarrow,n=\downarrow,v+\downarrow,\left(\epsilon^{\frac{1}{v}}\right)=\downarrow,v+\frac{1}{v}\downarrow,\epsilon$$

$$\therefore\quad v\downarrow,n=v\downarrow,v+\downarrow,\epsilon$$

$$\therefore\quad v\downarrow,n-\downarrow,\epsilon=v\downarrow,v$$

$$\text{or }\frac{n^v}{\epsilon}=v^v\quad\text{or }\frac{1}{\epsilon}=\left(\frac{v}{n}\right)^v$$

Taking the $\frac{1}{n}$ root of both sides of the last equation we obtain an equation easily solved by the Dual method.

$$\left(\frac{1}{\epsilon}\right)^{\frac{1}{n}}=\left(\frac{v}{n}\right)^{\frac{v}{n}}=z^z\text{, putting } z \text{ for } \frac{v}{n}$$

Hence we have the final equation

$$z^z=\left(\frac{1}{\epsilon}\right)^{\frac{1}{n}}\text{, or } z\downarrow,z=\downarrow,\left(\frac{1}{\epsilon}\right)^{\frac{1}{n}}=\frac{1}{n}\downarrow,\left(\frac{1}{\epsilon}\right).$$

Before the introduction of Dual arithemetic, an independent and direct solution of equations of this form defied the skill of mathematicians (see "Dual Arithmetic," Part II., pp. 91, 100.)

$$\downarrow,\left(\frac{1}{\epsilon}\right)='100000000\,;\ \text{and}\ \frac{1}{n}=\frac{1}{5}=\frac{b}{a}\,;$$

$$\therefore\quad z\downarrow,\ z='20000000.$$

$$\therefore\quad z=\frac{1}{12}\downarrow 0,'6\,2,5,6,8,3,3,=\cdot 078658307\,;\ \therefore v=\cdot 393291535,$$

found as at page 124. The value of $z,=\frac{v}{n}$, not attainable by any previously known method, may be almost instantly obtained under a variety of forms by the Dual calculus to any required degree of accuracy. In the present case,

$$z=\frac{1}{12}\downarrow '0'6\,2,5,6,8,3,3,1,7,\,;\ z=\frac{1}{20}\downarrow 4,7,2,1,9,8,2,2,=$$

$\cdot 078658307$, &c. Under the last form all the dual digits are positive.

$$\downarrow,\left(\frac{v}{n}\right)='254264202\quad\therefore\quad\downarrow,\left(\frac{n}{v}\right)=254264202,$$

$$\therefore\quad\frac{n}{v}=12\cdot 7132147;\ \text{and hence}\ \frac{1}{v}=2\cdot 54264294.$$

But we have before shown that $n=v\epsilon^{\frac{1}{v}}$ $\therefore\downarrow,\left(\frac{n}{v}\right)=$

$$\downarrow,\left(\epsilon^{\frac{1}{v}}\right)=254264202,\ \text{and}\ \downarrow,\left(\frac{1}{\epsilon^{\frac{1}{v}}}\right)='254264202\,;$$

Whence $s=\frac{v}{2}\left(\epsilon^{\frac{1}{v}}-\frac{1}{\epsilon^{\frac{1}{v}}}\right)$ becomes known.

$$s=\frac{v}{2}\,(12\cdot 7132147-\cdot 078658307)$$

$$=(\cdot 393291535\times 6\cdot 3172782)=2\cdot 48448448\,;$$

Because, $x+v+s=n=5$. ∴ $x=2\cdot 12222398$.

In these calculations, Q E $=y$ is put $=1$; but in the question, Q E $=113$ ft.

$\therefore$ $x = 239{\cdot}8113097$ ft. $=$ E O $=$ QF ; $s = 280{\cdot}74675$ ft. $=$ arc Q O; and $v = 44{\cdot}441966$ ft. $=$ F R $=$ O C $=$ the length of chain that is equal to the tension at O. $v + x = 284{\cdot}253254$ ft. $=$ Q R $=$ the length corresponding to the tension at Q or H.

Ques. 67. In the Chelsea Suspension Bridge, Thomas Page, Engineer, the central span between the piers is 348 ft. with a deflection O X, Fig. 111, of 29 ft.; supposing the entire weight of the chain A O B, when the weight of the platform, roadway, and full load is transmitted to it, to be 1·5 tons for each foot of its length, required the strains at the highest points A and B, and the tension at the lowest point O, as well as the length of the catenary curve passing through the points A, O, B?

$$\frac{348}{2} = 174\,;\quad \frac{29}{174} = \frac{1}{6} = c^2$$

$$\therefore\quad 2c = {\cdot}81649658.$$

Whence, equation (V.), becomes

$$\cdot 81649658\sqrt{z} = \epsilon^z - \frac{1}{\epsilon^z} = 2c\sqrt{z}\,;\qquad (a).$$

z being put for $\frac{1}{2v}$; v representing the length of the chain whose weight is equal to the tension at the lowest point O.

We propose to find the value of v under the form $\downarrow u_1\, u_2\, u_3 \ldots$

To find a convenient value for u_1, assume $z = \downarrow u_1$, then (a) gives

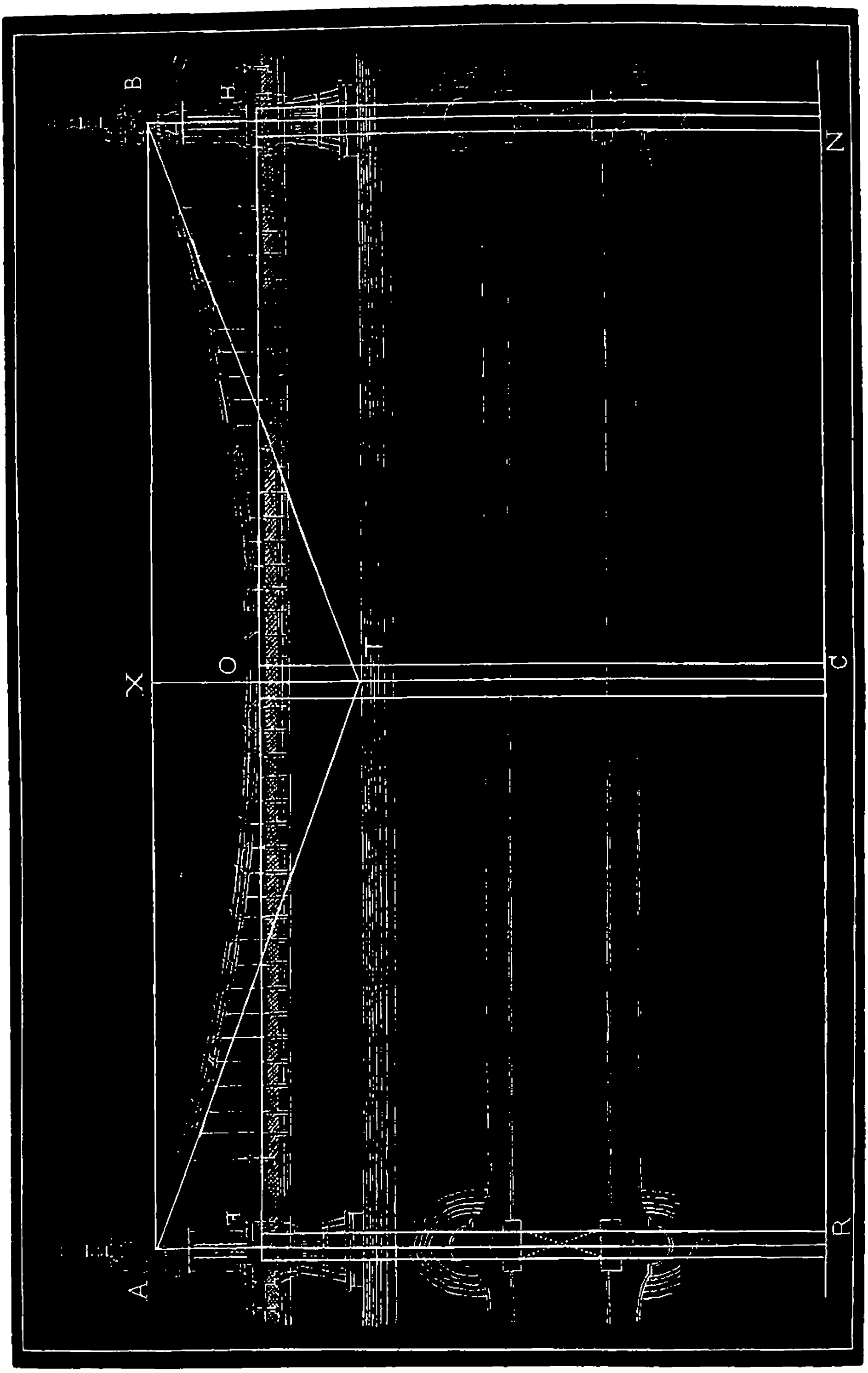
$$(1 \overset{+}{\downarrow} [u_1) - (1 \underset{+}{\uparrow} [{}'u_1) = 2c\sqrt{z} = 2c\sqrt{u_1 \times {\cdot}09531018}$$

9531018, being the dual of a logarithm of $\downarrow 1$, or of each unit in u_1. Since we arrive at the exact valve of z whether we assume u_1 too great or too small

$$(1 + [{\cdot}u_1) - (1 - [{\cdot}u_1 =) + 2\frac{u_1}{10} \text{ may be put} = 2c\sqrt{u_1 \times {\cdot}09531018}$$

$$\therefore\quad \frac{u_1{}^2}{100} = c^2\,(u_1 \times {\cdot}09531018) = \frac{1}{6}({\cdot}u_1 \times {\cdot}09531018).$$

Fig. 111.


B
H
N
X
O
T
C
A
F
R

$$\therefore\ u_1=\frac{9\cdot531018}{6}=1\cdot59$$

$\therefore$ The reciprocal of u_1, or $'u_1$ may be put$=\downarrow\,'1'5'9\ \ldots\ldots$

$$\text{or } u_1,=\downarrow 1,7,$$

$$\downarrow,\ (1\cdot17934889)=\downarrow,1,7,0,0,0,0,0,0,=16496249,$$

$$\text{Reciprocal}='16496249=\downarrow,'1'5'9'3'4'5'7'7=\downarrow,(\cdot84792554).$$

It is easily found that o, is the most convenient value for u_3, hence we may put in (a) $\underline{z=\downarrow 1,7,0,u_4,}$

$$\therefore\quad \epsilon^z-\frac{1}{\epsilon^z}=2c\sqrt{\cdot16496249+\cdot00010000u_4}$$

$$=2c\sqrt{\cdot16496249}\,(1+\cdot00030310u_4)$$

$$=\cdot33162478\,(1+\cdot00030310u_4);$$

$\epsilon^z-\dfrac{1}{\epsilon^z}$ may also be put under the form

$$1\cdot17934889\,(1\overset{+}{\downarrow}[u_4)-\cdot84792554(1\underset{+}{\uparrow}['u_4)$$

$$=(1\cdot17934889)\,(1+\cdot00010000\,u_4)$$

$$-(\cdot84792554)(1-\cdot00010000\,u_4)$$

$$=\cdot33142335+\cdot00020373\,u_4$$

$$\therefore\quad \cdot33142335+\cdot00020373u_4=\cdot33162478\,(1+\cdot00030310u_4)$$

$$=\cdot33162478+\cdot00010052\,u_4$$

$$\therefore\quad u_4=\frac{\cdot00020143}{\cdot00010052}=\frac{20143}{19052}=1\cdot952$$

$\therefore$ z may be again put$=\downarrow 1,7,0,1,9,u_6$, when a higher degree of accuracy is required.

$$\downarrow,(1\cdot17957297)=\downarrow,1,7,0,1,9,0,0,0,=16515249,$$

$$\text{Reciprocal, }'16515249=\downarrow,'1'5'9'5'3'6'2'1=\downarrow,(\cdot84776406)$$

$$\begin{array}{r}1\cdot17957297\\ \cdot84776406\\ \hline \cdot33180891\end{array}\qquad\qquad \begin{array}{r|l}1\cdot17957297 & \\ \cdot84776406 & \\ \hline 2\cdot02733 & 703\\ \hline \cdot00000 & 203u_6\end{array}$$

$$2c\sqrt{\cdot 16515249+\cdot 00000100u_6}=$$

$$2c\sqrt{\cdot 16515249}\ (1+\cdot 00000303u_6)$$

$$=\cdot 33181570\ (1+\cdot 00000303u_6)$$

$$\therefore\quad \cdot 33180891+\cdot 00000203u_6=\cdot 33181570+\cdot 00000101u_6$$

$$\therefore\quad u_6=\frac{\cdot 00000679}{\cdot 00000102}=\frac{679}{102}=6\cdot 66$$

$$\therefore\quad \downarrow,\ z=\downarrow,1,7,0,1,9,6,6,6,=16515915,$$

$$\text{or } \epsilon^{\cdot 16515915}-\frac{1}{\epsilon^{\cdot 16515915}}=(\cdot 81649658)\sqrt{\cdot 16515915}.$$

To find the value of z in such equations as (a), to any required degree of accuracy, and by such direct, independent, and simple means, is, without doubt, a great mathematical achievement.

$z=\dfrac{1}{2v}$ $\therefore$ $v=3\cdot 02738298$=three times the length of OC.

$x+v=3\cdot 19404964$; and the length of the catenary curve passing through the points A, O,$=\sqrt{x^2+2vx}$, (I.);

$\therefore$ the length of the curve A O=1·01828555. But when A X=174 ft., then $v=526\cdot 7646$ ft.; $x+v=555\cdot 7646$ ft.; and the length of the catenary curve passing through the points A, O, B,=354·36337 ft. (526·7646) × (1·5) = 790·1468 tons, the tension at the lowest point O in the direction of the tangent O F, and represented by the lengths of the three lines O C. (555·7646 × (1·5)=833·6469 tons, the tension at the highest point A, but in the direction of the tangent A T; this tension is represented by the lengths of the three lines F R, and the line A F. The sectional area at the centre of the bridge =214 sq. inches, and the sectional area at A and B=230 sq. inches; hence a weight of only 5 tons to the square inch gives a power at O of 1070 tons, and at A and B a power to sustain a tension of 1150 tons with ease.

CHAPTER VIII.

PRESSURE OF WATER AND OTHER FLUIDS.

LET L F B D, Fig. 112, be a vessel of any form whatever, filled with water, $a\,b$ any portion of the surface F K C B in contact with the water, G the centre of gravity of $a\,b$, G R the perpendicular depth of G below the surface of the water;

then if R G$=$10 ft., and the area of the surface $a\,b=3$ square feet, the lbs. pressure on this surface will be $=3\times10\times62{\cdot}5=$ 1875 lbs.

1875 lbs. is the weight of a column of water whose base is the area $a\,b$, and perpendicular height the depth of the centre of gravity of $a\,b$.

If the area of $c\,e$ on the bottom of the vessel$=$5 square feet, and Q $G_1=14$ ft. the perpendicular depth of the centre of gravity G_1 below the surface of the water; then the lbs. pressure on the area $c\,e=$

$$5\times14\times62{\cdot}5=4375 \text{ lbs.}$$

Again, if the area of the surface $h\,f$, on the slanting face A L$=$4 square feet, P $G_2=11$ ft., G_2 being the centre of gravity of the area $h\,f$; then the pressure on this slanting surface will be

$$4\times11\times62{\cdot}5=2750 \text{ lbs.}$$

As in former cases, the weight of a cubic of water is taken $= 62\frac{1}{2}$ lbs. It may be further observed that the positions or inclinations of the surfaces *a b*, *c e*, *h f*, are not taken into account, but merely their areas and the perpendicular distances of the centres of gravity from the horizontal surface of the fluid. On this simple principle rests that department of mechanics termed *hydrostatics;* it is easily demonstrated, as gravity acts on all the particles of the fluid, and each particle presses on that next below it, and, further, because, from the peculiar property of the fluid, this pressure is transmitted in all directions equally.

Ques. 68. Find the lbs. pressure on a floodgate whose breadth is 9 ft., and depth 7 ft.

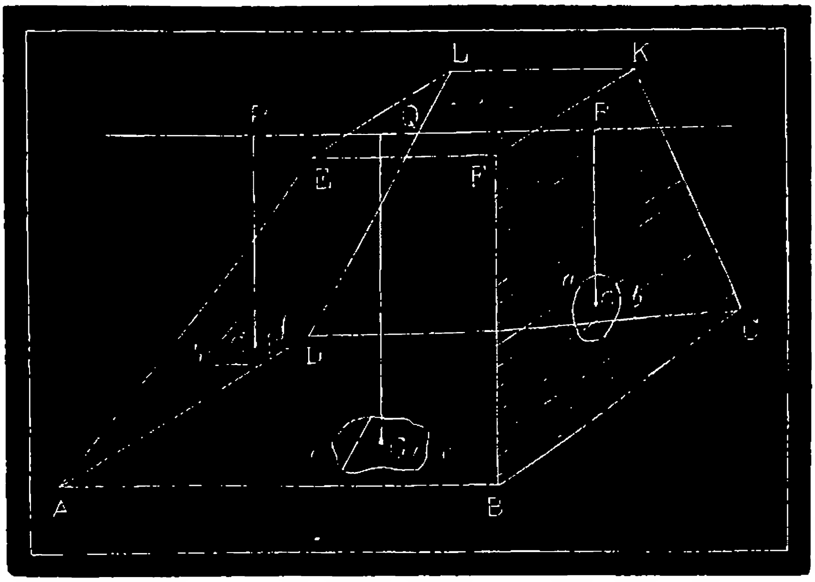
$$7 \times 9 = 63 \text{ square feet area,}$$

$$\text{depth of centre of gravity} = \frac{7}{2} = 3{\cdot}4 \text{ ft.}$$

$$\therefore \quad 63 \times \frac{7}{2} \times 62\tfrac{1}{2} = 13781{\cdot}25 \text{ lbs.}$$

Ques. 69. What is the pressure upon 10 ft. length of an embankment, the depth of the water pressing against it being 11 ft.?

Fig. 112.



$$10 \times 11 \times \frac{11}{2} \times 62{\cdot}5 = 37812{\cdot}5 \text{ lbs.}$$

Ques. 70. Required the relation of the pressure upon the four sides of a cubical vessel filled with water, and the pressure on the bottom which is horizontal?

Put n=the length of the side of the cube in feet;

$$n^3 \times 62{\cdot}5 = \text{pressure on the base;}$$

$$n \times n \times \frac{n}{2} \times 62{\cdot}5 = \text{pressure on one of the sides,}$$

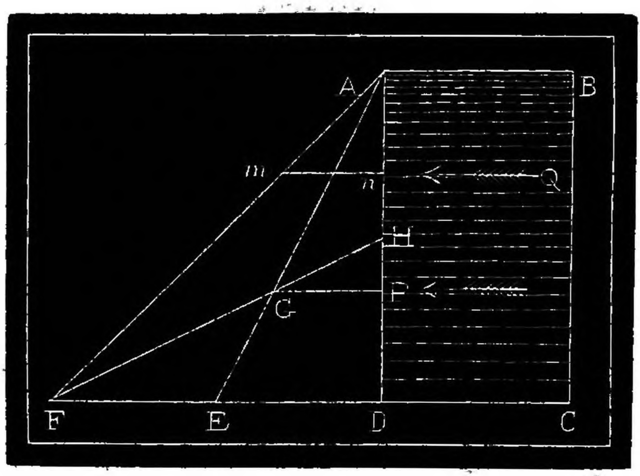
$$\therefore \quad 2\,n^3 \times 62{\cdot}5 = \text{pressure on the four sides.}$$

$$\text{Then } n^3 \times 62{\cdot}5 : 2\,n^3 \times 62{\cdot}5 :: 1 : 2,$$

that is, the pressure on the sides is = to twice the pressure on the base. In these calculations the fluid is supposed to be at rest, and acted on only by gravity.

Let A B C D be a vessel filled with water, the pressure Q on any point n, in the side A D, Fig. 113, is due to the perpendicular depth A n. If in the base D C produced we take D F=D A, the perpendicular depth of the water, then the pressure upon the point D will be due to the pressure of a column of the fluid, whose height is = D F. Draw F A, and from any point n draw $m\,n$ perpendicular to A D; hence $m\,n$=A n, and the pressure Q on the point

Fig. 113.



n will be due to a column whose height is $m\,n$; the same reasoning applies to any other point in the side of the vessel.

Let us take an example and compare this method of viewing the subject with the one previously enunciated and illustrated.

Ques. 71. What is the pressure on an embankment whose length is 21 ft. and depth of the water A D = 12 ft.?

The whole pressure upon the side of the embankment is equivalent to the pressure or weight of a mass of fluid of the form of a wedge, A F D, Fig. 113.

$$\text{Area of the triangle } A\,D\,F = 12 \times \frac{12}{2} = 72 \text{ square feet,}$$

$$\therefore \quad 72 \times 21 = 1512 \text{ cubic feet content of the wedge.}$$

$$\therefore \quad 1512 \times 62{\cdot}5 = 94500 \text{ lbs. pressure.}$$

Before it was stated that the pressure in lbs. on the side is equal to a column of water whose base is the area of the surface, and perpendicular height the depth of the centre of gravity.

It is evident, since the side of the embankment is a parallelogram, the depth of its centre of gravity

$$= \frac{12}{2} = 6 \text{ ft.,}$$

$$\text{area of the surface} = 12 \times 21 = 252 \text{ square feet.}$$

$$\therefore \quad 252 \times 6 \times 62{\cdot}5 = 94500 \text{ lbs., the pressure before found.}$$

It may be easily perceived that there is a certain point in the side A D of an embankment or vessel filled with water, where a single pressure will counterbalance the pressure of the water against the whole side. This point is called

THE CENTRE OF PRESSURE.

The *centre of pressure* must evidently lie in the line P G passing through the centre of gravity, G, of the *wedge of pressures*, of which the plane A F D is a cross section.

Bisect F D in E, and D A in H, draw A E and F H; these lines cut one another in the centre of gravity G. D P$=\frac{1}{3}$ D A, that is, the centre of pressure, P, in this case lies at $\frac{1}{3}$ of D A, from the bottom, see page 246.

Ques. 72. Required the pressure on the staves of a cylindrical vessel filled with water, the diameter of the base being 10 ft. and the perpendicular height 8 ft.?

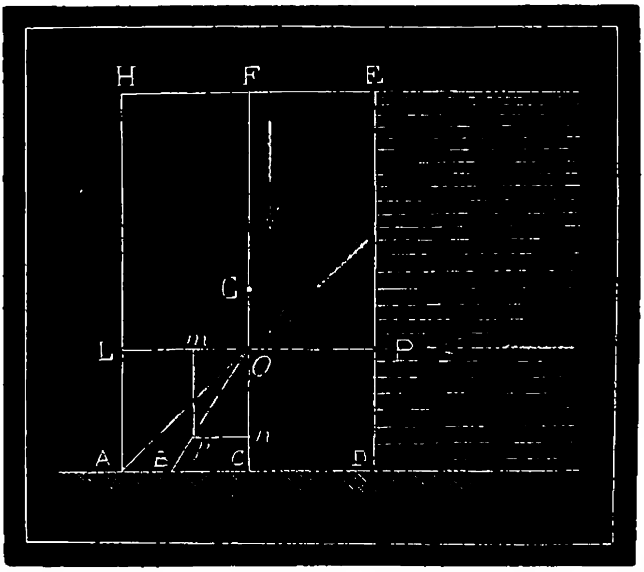
$3{\cdot}1416 \times 10 = 31{\cdot}416$ ft. the circumference of the cylinder,

$$\therefore \quad 31{\cdot}416 \times \frac{8}{2} \times 8 \times 62{\cdot}5 = 62832 \text{ lbs. pressure.}$$

If the staves of this barrel are to be kept together by a single hoop, that hoop should be $\frac{10}{3} = 3\frac{1}{3}$ ft. from the bottom.

Ques. 73. An embankment H D, Fig. 114, resists a pressure of water, whose *centre of pressure* is at P; it is required to determine by construction the conditions of equi-

Fig. 114.



librium, supposing when the pressure is sufficient to overturn the embankment it will turn upon A, as a centre?

Let F O C be the vertical line drawn through G, the centre of gravity of the embankment. Draw P L perpendicular to F C, intersecting F C in O. Make O n = the lbs. pressure in the embankment, and O m = the pressure of the water, complete the parallelogram O m p n, then if the diagonal O p or O p produced falls as at B inside the base, the embankment will stand, but if the diagonal cuts outside of A, embankment will fall by turning over upon O.

Otherwise, since the pressure of the water P, in pounds multiplied by the length of A L in feet, gives the moment of the water tending to turn the embankment, H D, on A as a centre; and the product of the weight of the embankment, H D, in pounds by the length of A C in feet, gives the momentum of the embankment that acts against the pressure of the water; consequently, when these moments are equal the embankment, A H E D, is upon the point of turning over the point A; if the moment of the water be the greater of the two, the structure will fall, but if it be the lesser of the two, it will stand.

Ques. 74. Suppose 10 ft. to be the length of an embankment whose height, D E, from the surface of the water at E, is 28 ft., A D = 6 ft., will the embankment stand or fall when a cubic foot of the material of which it is composed = 160 lbs.?

Surface upon which the water presses $= 28 \times 10 = 280$ square feet.

$$\text{Pressure of the water} = 280 \times \frac{28}{2} \times 62{\cdot}5 = 245000 \text{ lbs.}$$

$\frac{28}{3}$ = D P = A L = the distance of the centre of pressure P, from the bottom A D.

$$\therefore \quad 245000 \times \frac{28}{3} = 2286666\frac{2}{3} \text{ the moment of the water.}$$

Weight of the embankment $= 28 \times 10 \times 6 \times 160 = 268800$ lbs.

Moment of the embankment $=268800\times\frac{6}{2}=806400$.

This structure must fall, since the moment of the water is greater than the moment of the embankment.

Ques. 75. What must be the height of the water in the last question, so that the embankment may be upon the point of overturning?

Putting x for the required height, then the moment of the water will be

$$x\times 10\times\frac{x}{2}\times 62{\cdot}5\times\frac{x}{3}=\frac{x^3}{6}\times 625.$$

$$\therefore\quad \frac{x^3}{6}\times 625=806400.$$

$$\therefore\quad x=20{\cdot}59967 \text{ ft., height required.}$$

Ques. 76. Required the thickness of a rectangular embankment that supports a pressure of water rising its full height of 16 ft. when the structure is upon the point of turning over; the weight of a cubic foot of the material, of which the embankment is composed, 128 lbs.?

We may take the length of the embankment $=1$ foot, for if it stands for one foot of length, it will stand for any other length.

$$1\times 16\times\frac{16}{2}\times 62{\cdot}5=8000 \text{ lbs., the pressure of the water.}$$

$$\therefore\quad 8000\times\frac{16}{3}=\text{the moment of the water.}$$

If x be put for the thickness of the embankment, its moment will be

$$16\times x\times 1\times 128\times\frac{x}{2}=x^2\times 8\times 128.$$

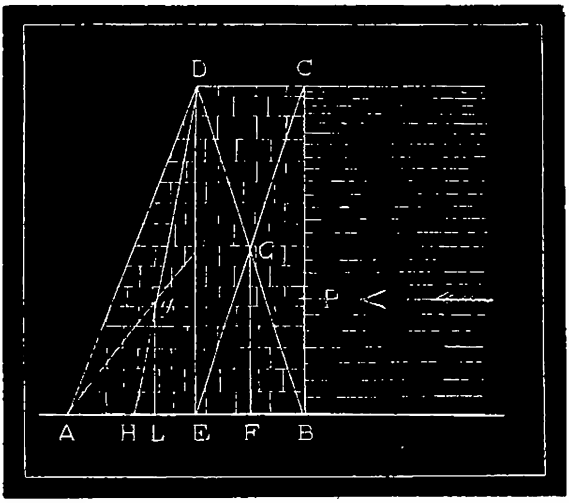
Putting $8\times 128\times x^2=8000\times\frac{16}{3}$,

Gives $x^2=\frac{125}{3}$; $\quad\therefore\quad x=6{\cdot}45497$ ft.

Ques. 77. Let the cross section of the embankment, A B C D, Fig. 115, have the form of a trapezoid, where A E = 6 ft., E B = 5 ft., B C = 15 ft., and the weight of a cubic foot of the material = 120 lbs.; as in former cases, the cubic foot of water is supposed to weigh $62\frac{1}{2}$ lbs.?

Let us consider the circumstances with respect to 8 ft. length of embankment, and suppose the cross section,

Fig. 115.



A B C D, to be divided into two parts, namely, the rectangular part, B C D E, and the triangular part, A E D. It has been before shown that a vertical line, *g* L, passing through the centre of gravity, *g*, of the triangular part, cuts the base, A L, supposed to be horizontal; so that $AL = \frac{2}{3}$ of $AE = 4$ ft. The vertical line, G F, passing through the centre of gravity, G, of the parallelogram, E D C B, cuts the base at F; so that

$$AF = AE + \tfrac{1}{2}EB = 8{\cdot}5 \text{ ft.}$$

It is supposed that the pressure, P, of the water tends to turn the embankment over a horizontal line passing through A, perpendicular to the plane of the paper.

$\frac{6 \times 1}{2} \times 8 \times 120 = 43200$ lbs. weight of the part of which A D E is a cross section; the moment of this part will be $= 43200 \times 4 = 172800$.

Weight of B C D E $=15\times5\times8\times120=72000$ lbs.

Moment of this part $=72000\times8{\cdot}5=612000$.

$\therefore$ $612000+172800=784800$, the moment of 8 ft. length of embankment.

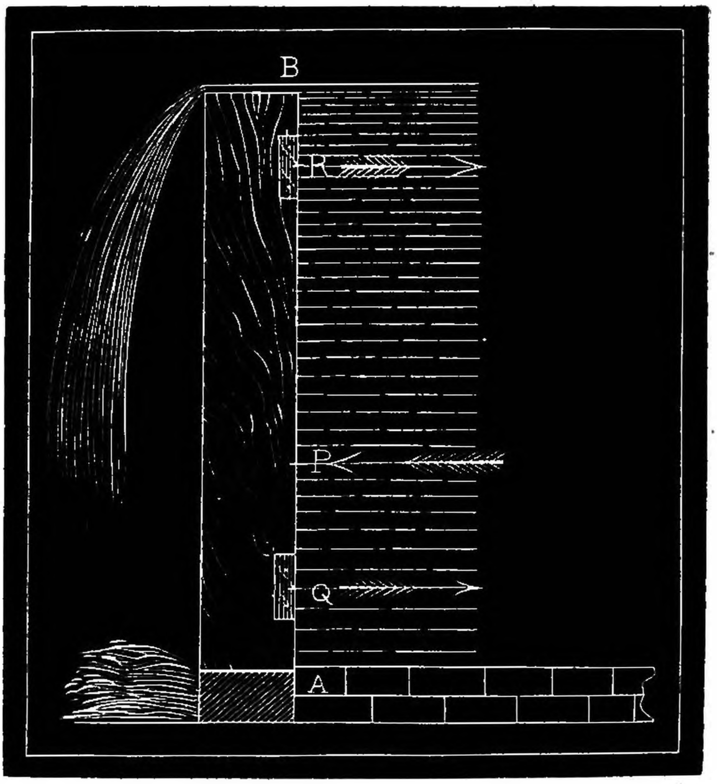
Since the moment of the water will be

$$15\times8\times\frac{15}{2}\times62{\cdot}5\times\frac{15}{3}=281250,$$

it follows that the embankment will stand.

Ques. 78. The breadth of a flood-gate is 12 ft.; the depth, A B = 8 ft.; the centre of the hinge, Q, is 18 in. from the bottom, A; and the hinge, R, is 18 in. from the surface, B; the pressure on Q, Fig. 116, is required?

Fig. 116



Since one-half the pressure of the water on the gate only acts on the hinges Q and R, that pressure in lbs. will be =

$$8 \times 6 \times \frac{8}{2} \times 62\cdot5 = 12000 \text{ lbs.}$$

Let P be the centre of pressure of the water, then

$$A\,P = \frac{8}{3}\ ;\ Q\,R = 8 - 3 = 5 \text{ ft.};$$

$$P\,R = P\,B - B\,R = \frac{8}{3} \times 2 - 1\tfrac{1}{2} = 3\tfrac{5}{6} \text{ ft.}$$

Because the pressure of the water at P is supported by the hinges at Q and R, then, on the principle of the lever (p. 33), supposing R to be the fulcrum,

$\therefore$ Putting x for the pressure on Q,

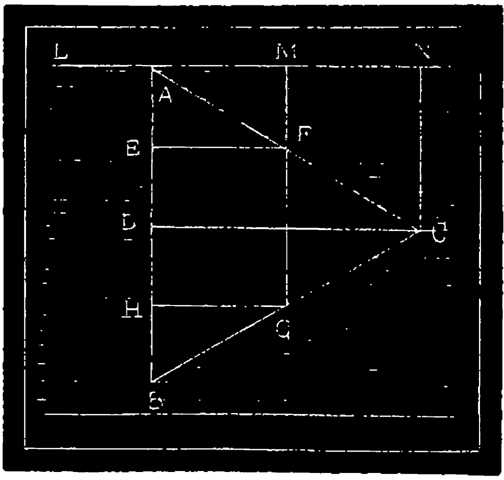
$$x \times Q\,R = P \times P\,R, \text{ that is}$$

$$x \times 5 = 12000 \times 3\tfrac{5}{6};$$

$$\therefore \quad x = 9200 \text{ lbs.}$$

Ques. 79. If one side of an equilateral triangle, immersed in a fluid, be perpendicular to the surface of the fluid, find the relation of the pressures on the three sides?

Fig. 117.



Let the side, A B, be perpendicular to the surface of the fluid L N, Fig. 117. From F and G, the points of bisection, and therefore the centres of gravity of A C, C B, draw E F, D C, H G, perpendicular to A B.

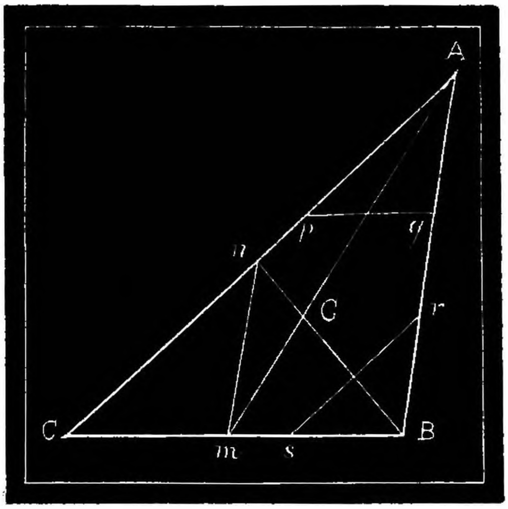
It is evident that the perpendicular depths, M F = A E, A D, M G = A H; of the centres of gravity of the sides, A C, A B, B C, are as

1 : 2 : 3.

Hence the pressure on the side B C is equal to the sum of the pressures on A B, A C.

Geometrical Proposition.

Fig. 118.



If from any of the angles of a triangle, A B C, a line, A m, be drawn to m, the middle of opposite side, C B, the point G is the centre of gravity of the triangle if $m\,G = \frac{1}{3}$ of $m\,A$.

Draw B n, bisecting A C, join m, n, then it is evident that all lines, as $p\,q$, parallel to C B are bisected by A m; hence the centre of gravity of the triangle must lie in A m. In the same manner it may be shown that B n bisects all lines, as $r\,s$, parallel to C A; therefore the centre of gravity is also in B n; consequently the point G, where A m and B n intersect, is the centre of gravity of the triangle A B C.

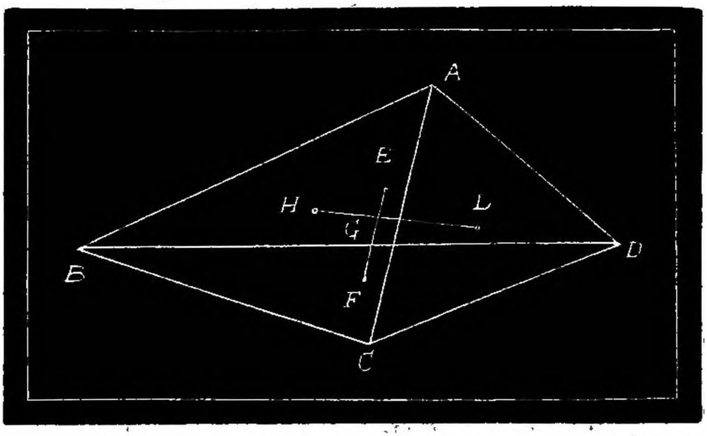
But $m\,n = \frac{1}{2}$ A B, and is also parallel to A B; and because the triangles $m\,n$ G and G A B are similar, $m\,G = \frac{1}{2}$ G A, whence $m\,G = \frac{1}{3}\,m\,A$.

As the knowledge of the position of the centre of gravity of a body is of much importance in almost every department of mechanics, and to save the trouble of distinct investigations in cases that often occur, we have thought it proper to add the succeeding results:

II. *The centre of gravity,* G, *of a trapezium,* A B C D, Fig. 119.

Let L be the centre of gravity of the triangle A D C, H of A B C, E of A B D, F of B D C; join H L and E F; these lines cut in G, the centre of gravity of A B C D.

Fig. 119.

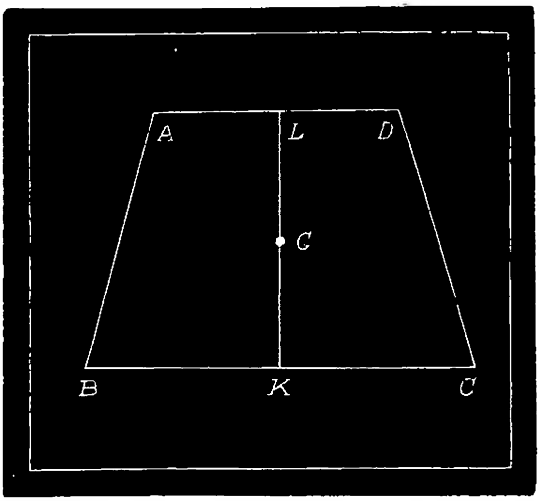


III. *To find the centre of gravity,* G, *of a quadrilateral,* A B C D, *when two sides,* A D, B C, *are parallel,* Fig. 120.

$a = AL = LD$, and $BK = KC = b$; $KL = 3c$.

$$KG = c\,\frac{b+2a}{b+a}.$$

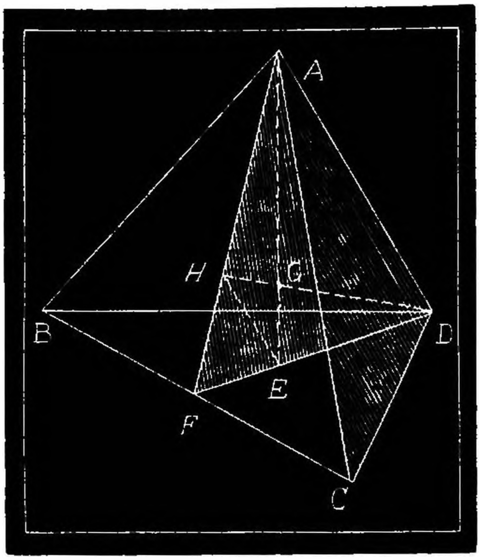
Fig. 120.



IV. *To find the centre of gravity,* G, *of any triangular pyramid,* A B C D, Fig. 121.

Put $AB = a$, $AC = b$, $AD = c$, and $BC = d$, $BD = e$, $CD = f$; then $AG = \frac{1}{4}\sqrt{3\,(a^2+b^2+c^2)-(d^2+e^2+f^2)}$.

Fig. 121.



Or, bisect B C in F, draw F D, F A; make E F$=\frac{1}{3}$ of F D, and H F$=\frac{1}{3}$ of A F, and draw H D, A E. The triangles H G E, G A D are similar.

$$\therefore \quad HG=\tfrac{1}{3}GD=\tfrac{1}{4}HD;$$
$$EG=\tfrac{1}{3}GA=\tfrac{1}{4}EA.$$

CASE II.

When B C$=$C D$=$D B.

$$\text{Then } AG^2=\tfrac{3}{16}(a^2+b^2+c^2-d^2).$$

CASE III.

When B C$=$C D$=$D B, and also A B$=$A C$=$A D.

$$\text{Then } AG^2=\tfrac{3}{16}\,(3a^2-d^2).$$

CASE IV.

If all the edges are equal, A B C D becomes a regular tetrædron.

$$\text{Then } AG=\tfrac{1}{4}a\sqrt{6}.$$

V. *To find the centre of gravity of a pyramid whose base is any polygon.*

The centre of gravity will be on the line drawn from the vertex to the centre of gravity of the base, and at the distance of $\frac{3}{4}$ of its length from the vertex.

VI. The centres of gravity of the surface of a cylinder, of a cone, and of a conic frustrum, are respectively the centres of gravity of the parallelogram, triangle, and trapezoid, which are vertical sections of the respective solids.

VII. The centre of gravity of the surface of a spherical segment is at the middle of its versed sine, or height.

VIII. In the cone, as well as in the pyramid, the distance of the centre of gravity from the vertex is $\frac{3}{4}$ of the axis.

IX. In a conical frustrum, the distance of the centre of gravity, measured on the axis from the centre of the less end

$$=\frac{h}{4}\,\frac{3R^2+2R\,r+r^2}{R^2+R\,r+r^2},$$

$h=$the height; R, r, the radii of the greater and lesser ends.

X. The last theorem will apply for the frustrum of any regular pyramid, taking R and r for the sides of the two ends.

XI. In a paraboloid, the distance of the centre of gravity from the vertex $=\frac{2}{3}$ of the axis.

XII. In the frustrum of a paraboloid, the distance of the centre of gravity from the centre of the lesser end, along the axis

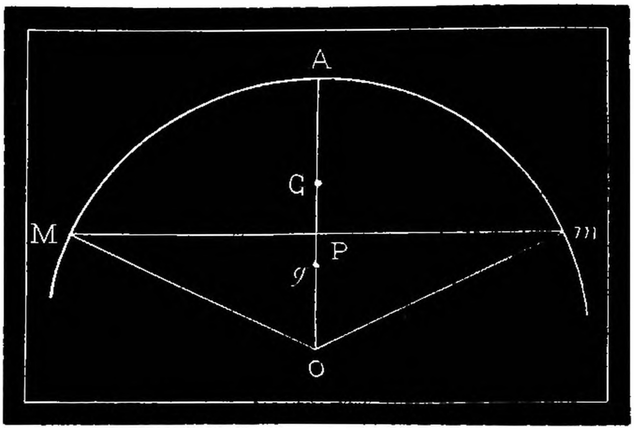
$$=\frac{h}{3}\,\frac{2R^2+r^2}{R^2+r^2};$$

h denotes the height, R and r the radii of the greater and lesser ends.

XIII. *To find the centre of gravity,* G, *of a circular arc* M A m, Fig. 122.

From the middle point of the arc A, draw A O to O, the centre of the circle; put $x=$A P, $y=$M P$=$P m, radius A O$=r$; the length of the half-arc, or M A$=z=$A m.

Fig. 122.



$$\text{Then } OG = \frac{ry}{z}.$$

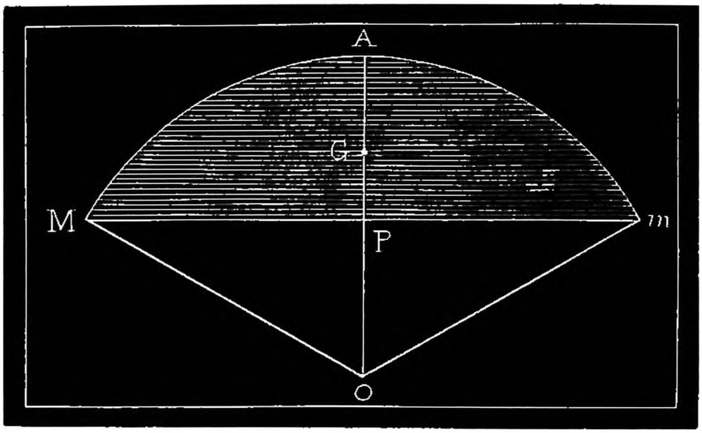
XIV. When the arc is a semicircle, then $y = 1$; and $z = \frac{1}{2}\pi = 1{\cdot}57079$,

$$\text{or } \frac{y}{z} = \frac{1}{1{\cdot}57079} = {\cdot}63662; \text{ and then}$$

$$OG = {\cdot}63662\, r.$$

XV. *To find the centre of gravity,* G, *of a circular segment,* M A *m*, Fig. 123.

Fig. 123.



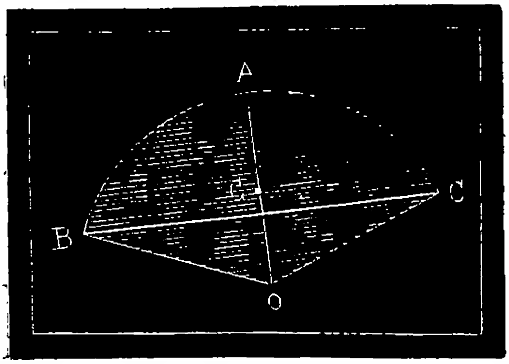
$$OG = \frac{MP^3}{3 \text{ area of } AMP}.$$

XVI. *When the circular segment becomes a semicircle.*

$$O\,G = \frac{r^3}{3 \text{ quadrants, radius } r} = \frac{r^3}{2{\cdot}356194\,r^2} = \frac{r}{2{\cdot}356194} =$$
$$\cdot 42441\,r.$$

XVII. In a circular sector, O B A C, Fig. 124, the distance from the centre of the circle to the centre of gravity, or O G, $= \frac{2cr}{3a}$; in which $r = $ O A; $c = $ B C; $a = $ the length of the arc B A C.

Fig. 124.



XVIII. To find the centre of gravity of a common parabola, Fig. 124.

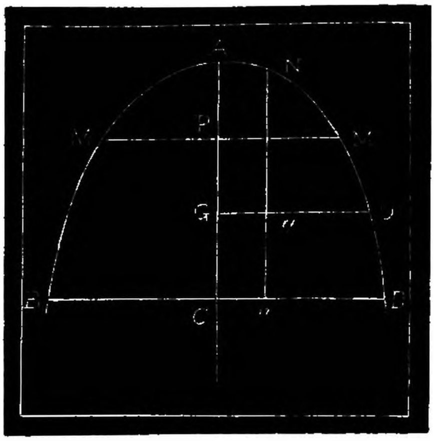
$$A\,G = \tfrac{3}{5}\,A\,C.$$

To find the centre of gravity, g, of the semiparabola, C A D, Fig. 125, take

$$G\,g = \tfrac{3}{8}\,C\,D.$$

XIX. *To find the centre of gravity,* G, *of the sector of a sphere,* O D A B, Fig. 126.

Fig. 125.



Let A C$=x$, O D$=r$, then

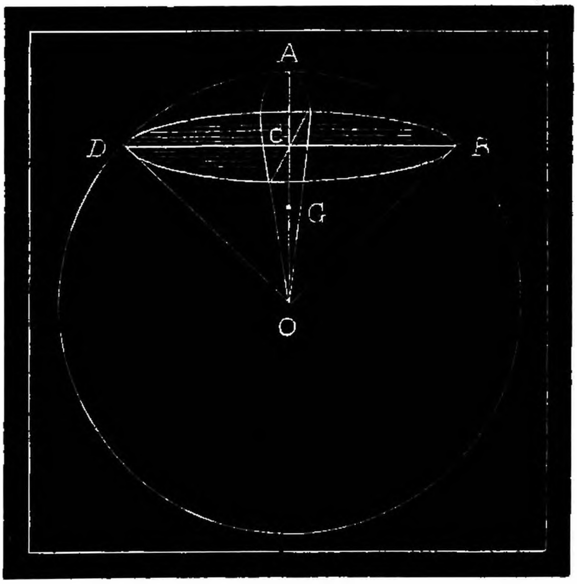
$$AG=\tfrac{1}{8}(2r+3x).$$

When $x=r$, the sector becomes a hemisphere, then

$$AG=\tfrac{5}{8}r.$$

XX. *To find the centre of gravity,* G, *of the segment of a spheroid.*

Fig. 126.



Let A be the vertex of the fixed axis a, putting c for the length of the revolving axis, then

$$AG=\frac{4a-3x}{6a-4x}\,x.$$

When the segment becomes a hemispheroid, then $x=\frac{a}{2}$ and $\frac{4a-3x}{6a-4x}x=\frac{5}{8}x$, for the distance of the centre of gravity from the vertex, $\therefore$ $\frac{3}{8}x$ is its distance from the centre of the base.

If $c=a$, the spheroid becomes a sphere, and, as the theorem is independent of c, it is alike applicable to both solids and their corresponding segments.

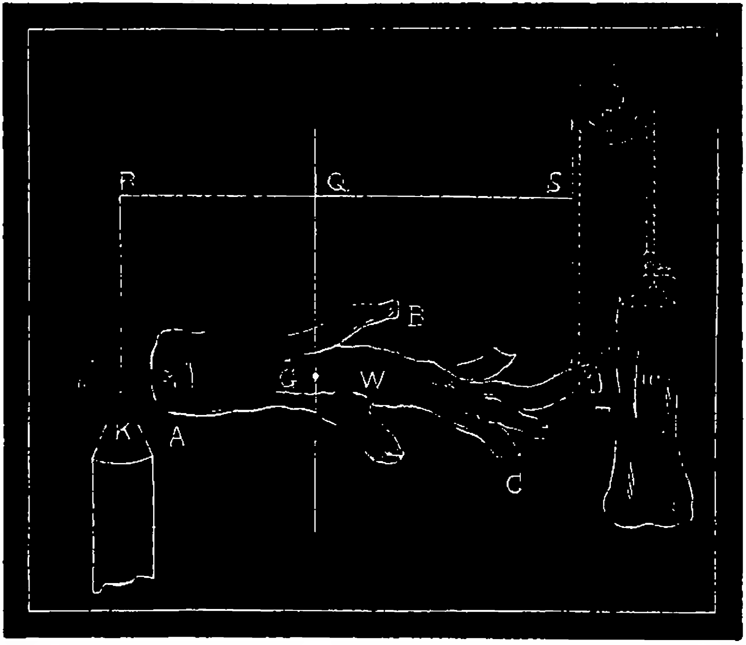
XXI. *To find the centre of gravity of a hyperboloid.*

Putting g = the distance of the centre of gravity G, from the vertex A, and taking $y^2=\frac{c^2}{a^2}(ax+x^2)$;

$$g=\frac{4a+3x}{6a+4x}x.$$

XXII. The position of the centre of gravity G, of any irregular body, A B C, may be determined when balanced in the manner represented in Fig. 127, and applying the proportion,

Fig. 127.



$$W : w :: a : z = RQ.$$

$$\therefore \quad z = \frac{aw}{W}.$$

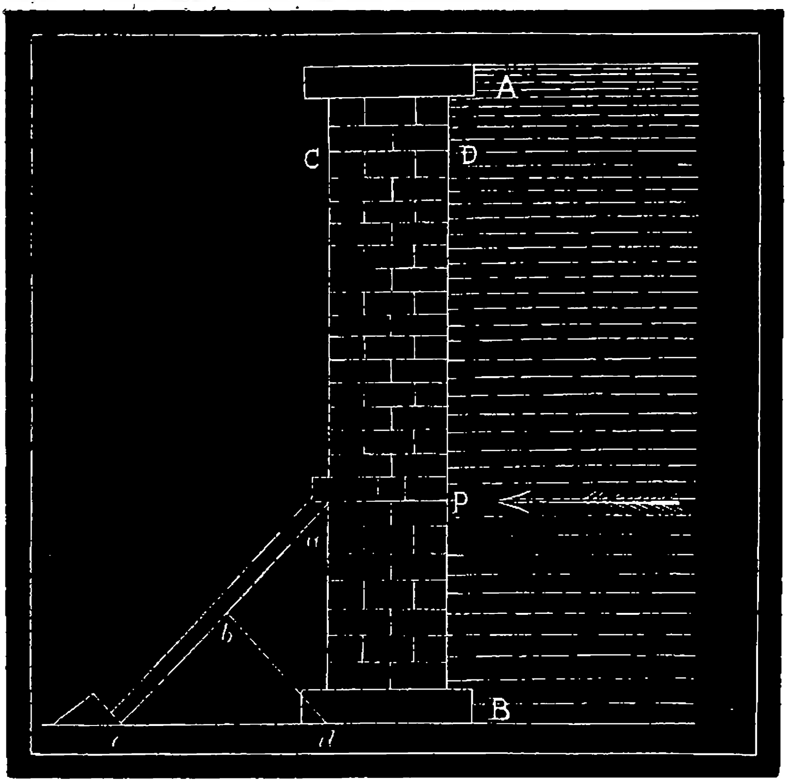
R S is put$=a$, and is horizontal.

K R, Q G, S T are perpendicular to S R.

Ques. 80. Let the shores, *a c*, support the wall, A C B, that sustains a pressure of water up to the top A; the stay delivers its thrust opposite the centre of pressure, P, of the water; the thrust on the stays is required, when the embankment is upon the point of turning over. Suppose $AB = 15$ ft.; $CD = 3$ ft.; the weight of a cubic foot of the material of which the wall is composed 120 lbs.; $ad = PB = 5$ ft.?

We will estimate for 8 ft. length of wall, but any other length may be selected.

Fig. 128.



$$\text{Weight of wall } 8\times3\times15\times120=43200 \text{ lbs.}$$

$$\text{Moment of the wall}=43200\times\tfrac{3}{2}=64800.$$

Let $b\,d$ be perpendicular to $a\,c$;

$$\therefore\quad b\,d=\sqrt{\frac{25}{2}}=\tfrac{5}{2}\sqrt{2}=3{\cdot}5355.$$

Put x for the thrust on the stays, $a\,c$, that prop 8 ft. length of wall; the moment of this thrust will be

$$x\times3{\cdot}5355.$$

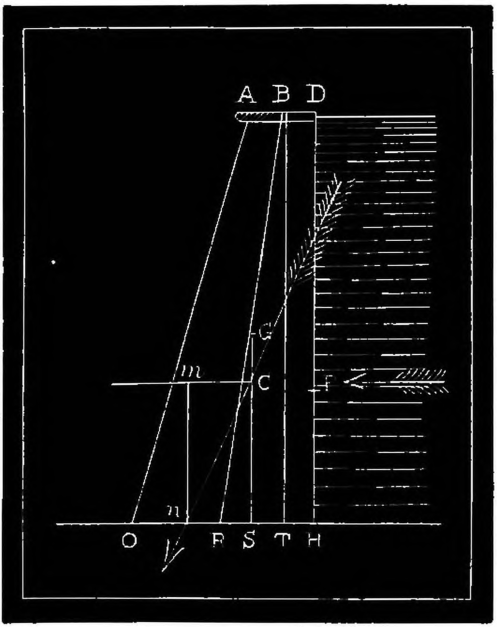
Moment of the pressure of the water will be

$$15\times8\times\frac{15}{2}\times62{\cdot}5\times\frac{15}{3}=281250.$$

$$\therefore\quad x\times3{\cdot}5355+64800=281250$$

$$\therefore\quad x=61222 \text{ lbs.}$$

Fig. 129.



Ques. 81. Required the modulus of stability of the stone structure, A D H O, Fig. 129. A D = 3 ft.; O H = 8 ft.; B R, drawn from the middle of A D to the middle of O H = 18 ft.; the height of the water, H D = 17·4 ft.; weight of a cubic foot of the material of which the wall is composed = 200 lbs.?

It is only necessary to investigate the action of the forces on a length of one foot. A D H O is a cross sector of the wall.

$$\mathrm{R\,G}=\tfrac{1}{3}\,\mathrm{B\,R}\;\frac{\mathrm{O\,R}+\mathrm{A\,D}}{\mathrm{O\,R}+\mathrm{A\,B}}=6\;\frac{4+3}{4+1\frac{1}{2}}=\frac{84}{11},$$

G being the centre of gravity of the wall.

$$RT = 4 - 1\tfrac{1}{2} = \tfrac{5}{2}$$

$$RB : RT :: RG : RS = 1\tfrac{2}{33}.$$

$$\therefore \quad SO = 5\tfrac{2}{33}.$$

$BT^2 = BR^2 - RT^2$, that is, B T is equal to the square root of $18^2 - (\tfrac{5}{2})^2 = \tfrac{1}{2}\sqrt{1271} = 17{\cdot}82554$.

Pressure of the wall, acting in the line G S, through the centre of gravity G=

$$\frac{8+3}{2} \times 1 \times 17{\cdot}82554 \times 200 = 19608{\cdot}094 \text{ lbs.}$$

The centre of pressure of the water is at P, and $HP = \frac{17{\cdot}4}{3} = 5{\cdot}8$ ft. $= CS$.

Pressure of the water $= 17{\cdot}4 \times 1 \times \frac{17{\cdot}4}{2} \times 65{\cdot}2 = 9211{\cdot}25$ lbs.

$$\therefore \quad 19608{\cdot}094 : 5{\cdot}8 :: 9211{\cdot}25 : 2{\cdot}724652 = Cm = Sn.$$

In the parallelogram $CSnm$, if the line $CS = 5{\cdot}8$, represents the pressure of the action in a vertical line passing through its centre of gravity; $Cm = 2{\cdot}724652$ represents the whole pressure of the water acting on P, its centre of pressure.

The ratio of S O to S n is termed the *modulus* of stability, which, in a good structure, should not be much less than $\tfrac{1}{2}$. In the present case,

$$\frac{Sn}{SO} = \frac{2{\cdot}724652}{5\tfrac{2}{33}} = {\cdot}5384, \text{ which is greater than } {\cdot}5,$$

Hence the structure is secure.

A square, A B C D, Fig. 130, is immersed vertically in a fluid, the side A B coinciding with the surface; if the diagonal, B D, be drawn, compare the pressure on the triangles A B D, B D C.

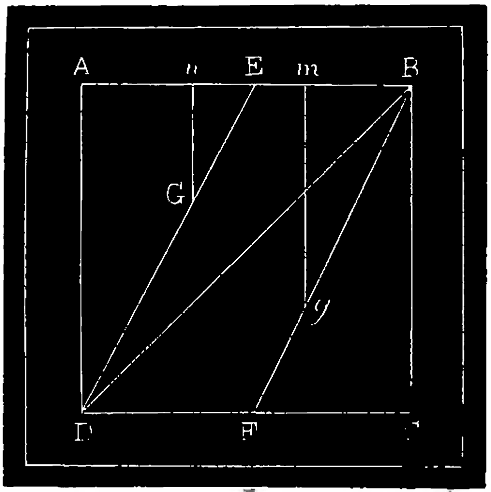
Bisect A B, D C, in E and F; join D E, B F; take $EG = \tfrac{1}{3}ED$, and $Fg = \tfrac{1}{3}BF$; G and g are the centres of gravity of the triangles A B D, B D C.

$$D\,E \text{ is equal and parallel to } B\,F,$$
$$B\,g = 2\,g\,F = 2\,E\,G;$$

∴ the perpendicular $m\,g$=twice the length of the perpendicular G n.

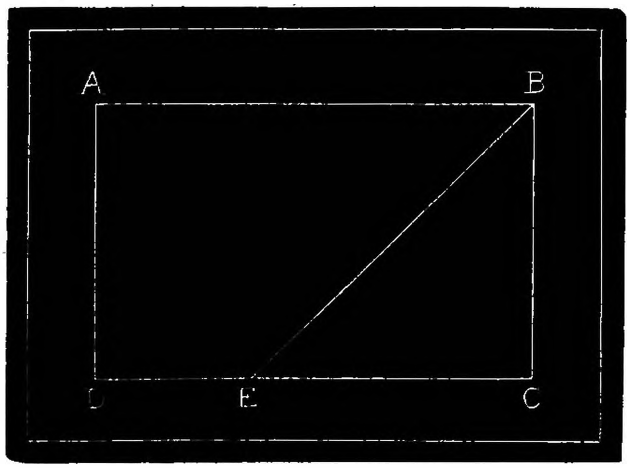
∴ The pressure on the triangle B C D is double the pressure on the triangle A B D. The same is true in the case of a rectangle, and the proportions remain the same whatever be the inclinations of the immersed planes, provided only that A B coincides with the surface of the fluid; for the perpendicular depths of the centres of gravity will be altered in the same ratio.

Fig. 130.



Given a rectangular parallelogram immersed vertically in water, with one side A B, Fig. 131, coincident with the

Fig. 131.



surface; it is required to draw from one of the angles, B, to the base a straight line, B E, so that the pressures on the parts A D E B, E B C, may be in the given ratio of m to n

It is evident that the pressure on the whole parallelogram is to the pressure on the triangle, so is $m+n$ to n.

Put $AB=a$; $AD=BC=b$, and $EC=x$;

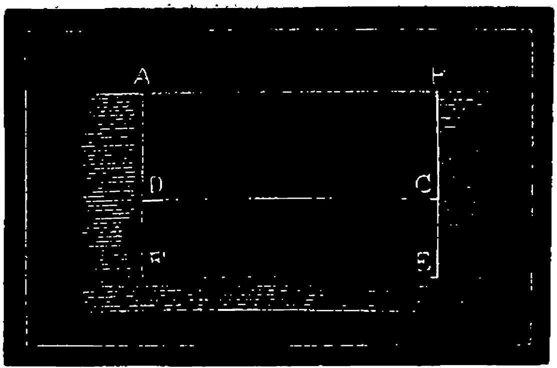
$$\therefore \quad a\times b\times\frac{b}{2} : \frac{1}{2}x\times b\times\frac{2b}{3} : m+n : n$$

$$\therefore \quad \frac{ab^2}{2} : x\frac{b^2}{3} :: m+n : n$$

$$\therefore \quad x=\frac{3}{2}\,\frac{n}{m+n}\,a.$$

To compare the pressures on the rectangles A C, C F, Fig. 132, immersed vertically in water, A B coinciding with the surface of the water.

Fig. 132.



The pressure on A B C D : that on A B E F

$$:: AD\times\tfrac{1}{2}AD : AF\times\tfrac{1}{2}AE$$
$$:: AD^2 : AF^2;$$

$\therefore$ The pressure A B C D : the pressure on D C E F :: $AD^2 : AF^2-AD^2$

Put $AF=b$ and $DA=x$, then $DF=b-x$.

When the pressure on A C is equal to the pressure on C F, then

$$x^2=b^2-x^2$$

$$\therefore \quad 2x^2=b^2;\ \text{or } x=\frac{b}{\sqrt{2}}.$$

$$\text{Or } 1 : \sqrt{2} :: x : b.$$

If the pressure on A B C D is to be to the pressure on D C E F, 5 to 7,

then

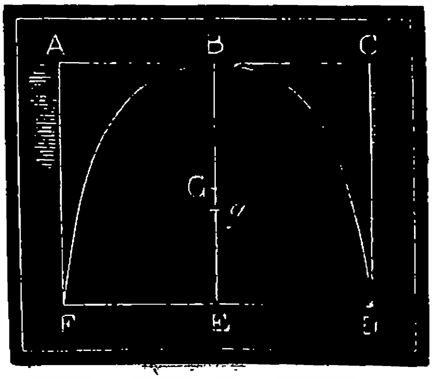
$$x^2 : b^2 - x^2 :: 5 : 7$$

$$\therefore \quad 7x^2 = 5b^2 - 5x^2$$

$$\therefore \quad x = \sqrt{\frac{5}{12}}\, b.$$

Let A C D F, Fig. 133, be a rectangular parallelogram immersed in water, the side A C coinciding with the surface, A B $= a$; A F $= b$. F B D is the inscribed parabola; find the ratio of the pressure on the parallelogram and the pressure on the parabola.

Fig. 133.



If G be the centre of gravity of A C D F, then

$$\text{B G} = \frac{b}{2} \text{ and the pressure} = a\,b \times \frac{b}{2} \times 62\tfrac{1}{2} = \frac{a\,b}{2} \times 62\tfrac{1}{2}.$$

The area of the parabola $= \frac{2}{3}$ of the parallelogram;

B $g = \frac{3}{5}\,b$, if g be the centre of gravity of the parabola;

$$\therefore \quad \frac{2a\,b}{3} \times \frac{3b}{5} \times 62\tfrac{1}{2} = \frac{2a\,b^2}{5} \times 62\tfrac{1}{2} = \text{the pressure on the parabola}$$

$$\therefore \quad \frac{ab}{2} \times 62\tfrac{1}{2} : \frac{2ab^2}{5} \times 62\tfrac{1}{2} :: \tfrac{1}{2} : \tfrac{2}{5}$$

$\therefore \quad 5 : 4$, is the ratio of the pressure on the parallelogram to the pressure on the parabola.

Show that if a hollow sphere be filled with a fluid, whose specific gravity is s, that the whole pressure against the internal surface is three times the weight of the contained fluid.

Let r=the radius; then $4\pi r^2 \times$ internal surface, π as usual$=3{\cdot}14159265$.

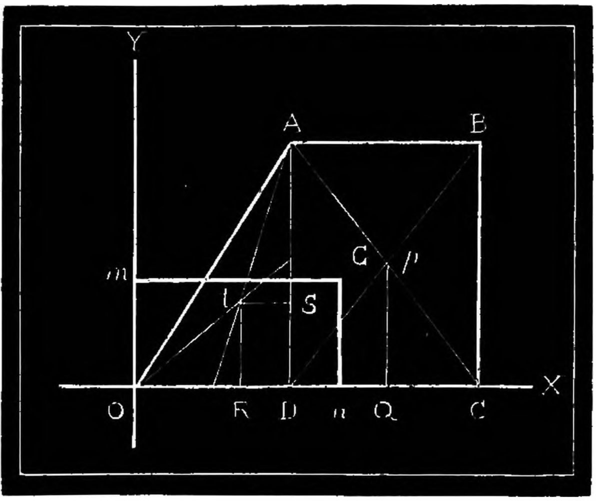
$\therefore$ the pressure$=4\pi r^2 \times r \times s = 4\pi r^3 s$;

The solid content of the sphere $= \frac{4}{3}\pi r^3$, and its weight $= \frac{4}{3}\pi r^3 s$.

$$4\pi r^3 s : \tfrac{4}{3}\pi r^3 s :: 1 : \tfrac{1}{3} \text{ or } 3 : 1.$$

Let G be the centre of gravity of a trapezoid A B C O, Fig. 134, A B$=a$, and parallel to O C$=b$; B C$=c$, and perpendicular to both A B and O C. The straight line O Y is drawn perpendicular to O C X, it is required to find general expressions for the perpendicular co-ordinates $x=m$ G$=$O n, and Y$=n$ G$=$O m.

Fig. 134.



Let p be the centre of gravity of the rectangle A B C D,

$$p\text{ Q}=\frac{c}{2} \text{ and Q O}=b-\frac{a}{2}=\frac{2p-a}{2}.$$

If t be the centre of gravity of the triangle O A D, then the co-ordinates of t will be, t R$=\frac{1}{3}$ A D$=\frac{1}{3}$ B C$=\frac{c}{3}$.

O R = ⅔ of O D $=\frac{2(b-a)}{3}$.

Area of A B C O × G n = Area of A B C D × p Q + area of A O D × t R; that is

$$c\frac{a+b}{2}y=ac\times\frac{c}{2}+\frac{(b-a)c}{2}\times\frac{c}{3},$$

$$\therefore\quad y=\frac{c}{3}\left(1+\frac{a}{a+b}\right).$$ See pp. 37, 38, 39.

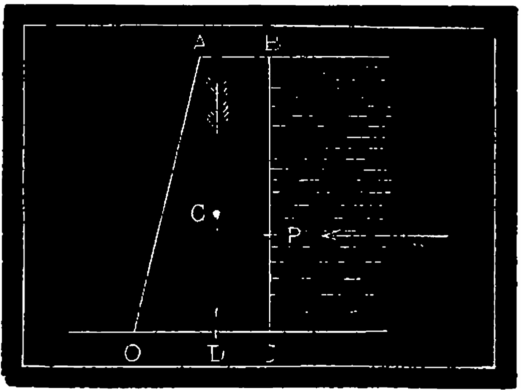
Again the area O A B C × G m = Area of A B C D × O Q + Area of A O D × R O; that is

$$c\frac{b+a}{2}x=ac\times\frac{2b-a}{2}+\frac{c(b-a)}{2}\ \frac{2(b-a)}{3}$$

$$\therefore\quad \frac{b+a}{2}x=a\frac{2b-a}{2}+\frac{(b\times a)^2}{3}$$

$$\therefore\quad x=\frac{2b}{3}-\frac{a^2}{3(a+b)}.$$ See pp. 37, 38.

Fig. 135.



Ques. 82. Given in A B C O, Fig. 135, which represents the cross section of an embankment made of brickwork, a cubic foot of which weighs 112 lbs.; A B = 1 ft., parallel to O C = 2 ft.; find the height, B C, which is perpendicular to both A B and O C, when the embankment is upon the point of overturning upon the edge at O by the pressure of the water which stands at the brim, B?

Put v = B C, the required height, then the pressure of the water on 1 ft. length of embankment =

$$v\times1\times\frac{v}{2}\times62{\cdot}5=\frac{v^2}{2}\times62{\cdot}5.$$

$$\text{Moment of the water}=\frac{v^2}{2}\times 62\cdot5\times\frac{v}{3}=\frac{v^3}{6}\times 62\cdot5.$$

∴ If G be the centre of gravity of the trapezoid A B C O, then by the last proposition,

$$\text{O D}=\frac{2}{3}\times 2-\frac{1}{3(1+2)}=\frac{11}{9}.$$

Pressure of a length of one foot of the embankment=

$\frac{1+2}{2}v\times 112=168\,v$ lbs.; moment of embankment=

$$168\,v\times\frac{11}{9}.$$

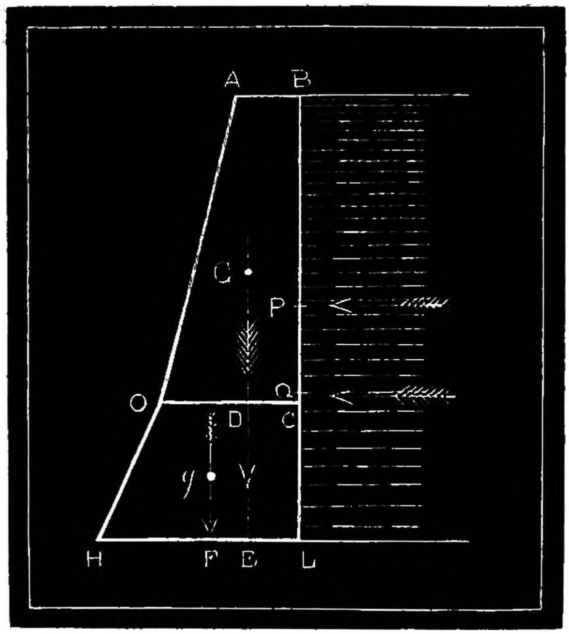
$$=\quad\frac{v^3}{6}\times 62\cdot5=168\,v\times\frac{11}{9};$$

$$\therefore\quad v\times\sqrt{19\cdot712}=4\cdot439821.$$

Weight of embankment$=168\,v=745\cdot89$ lbs.

Ques. 83. Let the embankment A B C O, Fig. 136, be the same as in the previous question; now if the embank-

Fig. 136.



ment be raised until H L=3 ft., what must be the perpendicular depth B L, so that when the part A B C O be upon the point of overturning at O, at the same time the whole embankment, A B L H, will be upon the point of turning over the edge through H?

$$\text{E L}=\text{D C}=2-\frac{11}{9}=\frac{7}{9}$$

$$\therefore \quad \text{H E}=3-\frac{7}{9}=\frac{20}{9}$$

$\therefore$ the moment of O A B C round the point H, will be $745{\cdot}89\times\frac{20}{9}$. If g be the centre of gravity of O C L H, then

$\text{H F}=\frac{2}{3}\times 3-\frac{4}{3(2+3)}=\frac{26}{15}$; putting z for B L, the moment of the part O C L H round H as a fulcrum will be

$$\frac{2+3}{2}(z-v)\,(112)\times\frac{26}{15}.$$

The moment of the water acting in opposition will be

$$z\times 1\times\frac{z}{2}\times 62{\cdot}5\times\frac{z}{3}=\frac{z^3}{6}\times 62{\cdot}5$$

$$\therefore \quad \frac{z^3}{6}(62{\cdot}5)=\frac{2+3}{2}(z-4{\cdot}43982)\,(112)\frac{26}{15}+745{\cdot}89\times\frac{20}{9}.$$

$$\therefore \quad z^3-46{\cdot}592\,z=-47{\cdot}73689344\,; \quad (\beta).$$

$$\therefore \quad z=6\downarrow 0,4,0,'5'1'9'9'5=6{\cdot}24037812.$$

By dual arithmetic in any cubic equation the value of the unknown quantity, as z, may be found in a few minutes under an endless variety of forms, but all reducible to one or other of the roots; in the present case z may be found under the form $z=6\downarrow 0,3,9,4,3,4,8,5$, which is also$=6{\cdot}24037812$

z may be found under the form $5\downarrow 2,3,1,1,3,2,0,1,=6{\cdot}24037812$,

When we assume z of the form $n\downarrow u_1\,u_2\,u_3\,\ldots\ldots$ a convenient value for n is readily found by supposing A B H L a trapezoid, then B L must be greater than 6, hence n was put$=6$; but we might assume $n=4$, 5, 6, or 7, and yet determine

$$z=6{\cdot}24037812.$$

The height and dimensions of any number of trapezoids forming an embankment like A L H, of equable strength, can be found in a similar way. A B, O C, H L, &c., may be given any common difference. The labour and difficulty experienced, before dual arithmetic was invented, in finding the roots of equations higher than a quadratic prevented writers on practical mathematics from introducing problems involving cubic or higher equations. The student should pay particular attention to these remarks.

In solving equation (β) we put it under the form

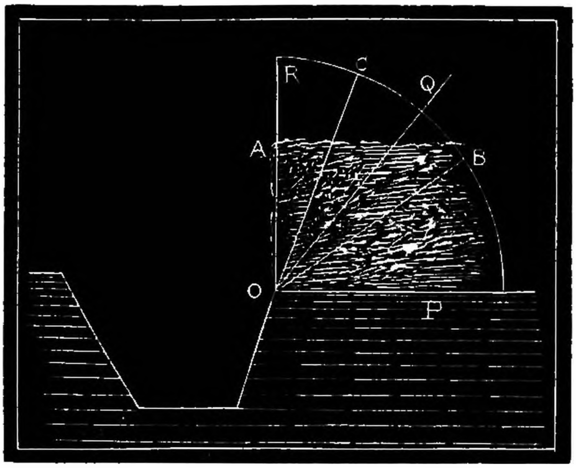
$$z^2+\frac{47{\cdot}73689344}{z}=46{\cdot}592.$$

Because, in the dual calculus, z^2 and $\frac{1}{z}$ become known at one and the same time. This arrangement being made, it is soon found that $z=6 \downarrow 0,4,0,'5\ 'u_5'u_6'u_7'u_8$.

REVETMENT WALLS.

If dry or wet clay, mixed earth, shingle, gravel, peat, &c., be excavated, and placed unprotected by a wall, or other support, the face, O A, in a perpendicular position, O P, Fig. 137, horizontal, during the wet and winter season, a portion of the embankment, A O Q, will slide away, and ultimately assume a permanent slope, as O Q; then O Q is termed the natural slope. The weight and natural slopes of different kinds of earths are given in most engineering pocket-books; but the natural slope of earth in engineering works should, if possible, be found in each particular undertaking. The thrust of earth upon a revetment wall is given by a portion, R O C, of the wedge R O Q; the line O C is termed the line of rupture. Coulomb found that the angle, R O C, which the line of rupture makes with the vertical, is one-half the angle R O Q, which the line of natural slope, O Q, makes with the vertical, O A.

Fig. 137.



To find the pressure of earth when it is level with the top of the revetment wall.

Rule. Multiply the weight of a cubic foot of the earth by the square of the tangent of the angle R O C (equal half R O Q), the product is to be taken as the weight of a cubic foot of a fluid pressing like water. It may aid the memory to observe, that the product above mentioned takes the place of the 62·5 lbs., the weight assumed for a cubic foot of water.

Ques. 84. A revetment wall, 36 ft. high and 8 ft. thick; any unit of length may be taken for convenience, we select a foot; this wall has to support the pressure of a mixed earth, the natural slope of which was found to be 34 degrees, and the weight of a cubic foot = 112 lbs.; the wall is built of sandstone, a cubic foot of which weighs 144 lbs.; will it stand or fall?

$\frac{1}{2}$ of 34°=17°; tangent of 17°=·3057307, taken from a table of natural tangents. ·3057 will be near enough for our present purpose.

·3057 squared=·09345249, of which we take ·09345, also near enough for our present purpose.

$·09345 \times 112 = 10·4664$ lbs., the weight of a cubic foot of the supposed fluid, acting at the back of the wall like water.

$$36 \times 1 \times \frac{36}{2} \times 10·4664 = \text{lbs. pressure of the earth.}$$

$$\text{Moment of the earth} = 36 \times 18 \times 10·4664 \times \frac{36}{3} = 81386·7264.$$

$$\text{Moment of the revetment} = 36 \times 8 \times 1 \times 144 \times \frac{8}{2} = 82944.$$

Hence, the wall will stand, but it is not very secure, for 82944 is not much greater than 81386·7264.

Ques. 85. When the wall is of brickwork, 16 ft. high, supporting damp clay, whose natural slope is 45°; find the thickness of the wall when it is no more than sufficiently strong to resist the pressure of the clay?

Natural tangent of 22° 30′=·4142136.

·4142 squared, ·1716; a cubic foot of brickwork=112 lbs.; damp clay weighs 120 lbs.; $·1716 \times 120 = 20·592$.

$\therefore \quad 16 \times 1 \times \frac{16}{2} \times 20·592 \times \frac{16}{3} = 14057·472$, the moment of the clay. Putting x for the thickness of the wall, we have the moment of the wall equal

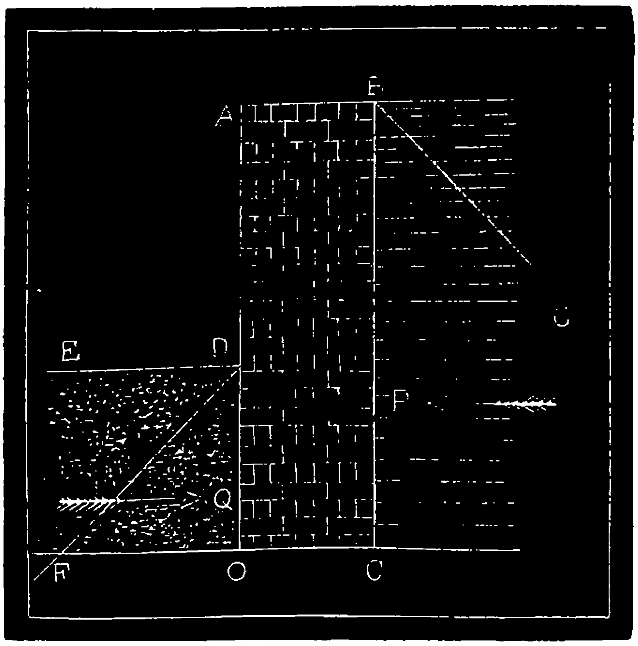
$16 \times 1 \times x \times 112 \times \frac{x}{2} = 896\,x^2$, which must be equal to 14057·472, when the wall is upon the point of overturning,

$$\therefore \quad x = \sqrt{\frac{14057·427}{896}} = 3·96095.$$

$\therefore$ The required thickness of the wall=4 ft., nearly.

Ques. 86. Let O A = 9·6 ft., O C = 4 ft., the weight of a cubic foot of the brickwork in A C = 112 lbs., the water is up to H, the embankment is supported by a bank of earth, O D, the natural slope of which is 45°, O D = 5 ft., E D is level; will the embankment stand or fall, E D O being composed of gravel and shingle, a cubic foot of which weighed 96 lbs.?

Fig. 138.



In making trial calculations, when a wall is supported by earth, as in Fig. 138, the weight of the equivalent fluid may be taken = 6 times the weight of the earth. On this supposition the earth, E D O, may be considered a fluid, a cubic foot of which = 96 × 6 = 576 lbs.

Then, $9·6 \times 1 \times 4 \times 112 \times \frac{4}{2} = 8601·6$, moment of the wall;

As usual, one foot is taken for the length.

$5 \times 1 \times \frac{5}{2} \times 576 \times \frac{5}{3} = 12000$, moment of the earth.

$9·6 \times 1 \times \frac{9·6}{2} \times 62·5 \times \frac{9·6}{3} = 9216$, moment of the water.

Since 8601·6 + 12000 is greater than 9216, the wall, O C A B, will stand.

Ques. 87. A revetment wall, A C, Fig. 138, is 30 ft. high = B C; O C = 6 ft. On one side, C B, earth of mean quality, 45° natural slope, is sustained level with the top, and on the other side, A O, the earth has a natural slope, F D, and rises to the height O D = 5 ft.; will the wall stand or fall, supposing that the weight of a cubic foot of the earth to be 120 lbs., and that of the wall 130 lbs.?

When a wall resists the pressure of earth with a natural slope at the top, represented by the line B G, Fig. 138, the weight of the equivalent fluid is roughly taken $= \frac{1}{8}$ the weight of the earth; but when the earth with a natural slope, D F, resists the pressure of the revetment, A C, and water or earth pressing on B C, then the weight of the equivalent fluid is roughly estimated at $\frac{1}{2}$ the weight of the earth, F D O.

In this case, the moment of the earth, which rises to the level of the wall, is opposed by the sum of the moments of the wall, and the earth on the other side having a natural slope.

Weight of the wall

$$=30 \times 1 \times 6 \times 130 = 23400 \text{ lbs.}$$

Moment of the wall

$$=23400 \times \frac{6}{2} = 70200;$$

Weight of a cubic foot of the equivalent fluid, with respect to the sloping earth

$$=120 \times \tfrac{1}{2} = 60 \text{ lbs.}$$

Pressure of this earth

$$=5 \times 1 \times 60 \times \frac{5}{2} = 750 \text{ lbs.}$$

$$\text{Moment} = 750 \times \frac{5}{3} = 1250;$$

Total moments sustaining the wall

$$=70200+1250=71450.$$

Moment of the level earth

$$=30\times 1\times\frac{30}{2}\times 120\times{\cdot}1716\times\frac{30}{3}=92664;$$

Since the latter moment is greater than the sum of two the former, it may be concluded that the structure will fall.

Referring to Fig. 137, page 265, and putting θ for the angle R O Q, then because $\tan^2\frac{\theta}{2}=\frac{1-\cos\theta}{1+\cos\theta}$, and may be employed when a table of natural tangents is not convenient.

The amount of pressure of a bank of earth, supposed to act in plane parallel layers against a vertical plane a foot in breadth=

$$\frac{h^2}{2}\left(\frac{1-\cos\theta}{1+\cos\theta}\right),\ h \text{ being the height.}$$

When the bank is composed of layers each sloping parallel to the natural slope, then the amount of pressure of the bank=

$$\frac{h^2\sin\theta}{2}.$$

But if the earth is made up of layers sloping from the vertical, O R, at an angle, R O B $=\phi$, greater than the natural slope, R O Q $=\theta$, the pressure of the earth=

$$\frac{h^2}{2}\sin\phi\,\frac{\sin\phi-\sqrt{\sin^2\phi-\sin^2\theta}}{\sin\phi+\sqrt{\sin^2\phi-\sin^2\theta}}.$$

MEASUREMENT OF HEIGHTS BY OBSERVATIONS OF THE BAROMETER AND THERMOMETER.

Much of what has been said respecting the pressure of water applies to all fluids, both liquids and gases; air and other elastic fluids have some properties peculiar to themselves; however, the nature and properties of both atmospheric air and mercury have been carefully ascertained. So far, the determination of the heights of mountains by the barometer presents no difficulty; but to solve the ultimate equation defied the skill of mathematicians, and the formulæ presented by writers on mechanics to effect this object only gave approximate results. This ultimate equation, which may be presented under the form (I.), can be solved with the greatest ease by dual arithmetic.

$$z = \frac{k(1+\alpha\tau)}{g}\left(1+\frac{z}{r}\right)\log_{\epsilon}\left[\frac{H}{h}\left(1+\frac{z}{r}\right)^2\right]; \qquad \text{(I.)}$$

In this equation all the quantities except the required height z are known. The logs. are hyperbolic, and (I.) may be put under the form

$$\sqrt{\frac{H}{h}}\left(1+\frac{z}{r}\right) = \sqrt{\frac{H}{h}} +$$

$$\frac{k(1+\alpha\tau)}{rg}\sqrt{\frac{H}{h}}\left(1+\frac{z}{r}\right)\log_{\epsilon}\left[\frac{H}{h}\left(1+\frac{z}{r}\right)^2\right].$$

In this last equation put $x = \sqrt{\frac{H}{h}}\left(1+\frac{z}{r}\right)$; $B = \frac{k(1+\alpha\tau)}{rg}$;

and put $\sqrt{\frac{H}{h}} = A$; and suppose $A =$ hyperbolic log. of a, that is, $\log_{\epsilon} a = A$. These substitutions being made, (I.) becomes

$$x = A + Bx \log_{\epsilon} x^2; \qquad \text{(II.)}$$

$$\epsilon = 2{\cdot}71828183, \text{ and } \downarrow' \epsilon = 10^8.$$

(II.) may be written $\epsilon^x = a x^{2Bx}$; taking the xth root of both sides,

$$\epsilon = a^{\frac{1}{x}} x^{2B}. \quad \text{Put } n^{2B} = \epsilon, \text{ then}$$

$$\frac{n^{2B}}{x^{2B}} = a^{\frac{1}{x}};$$

$$\therefore \quad \left(\frac{n}{x}\right)^x = a^{\frac{1}{2B}}, \text{ or } \left(\frac{x}{n}\right)^x = \frac{1}{a^{\frac{1}{2B}}}.$$

$$\therefore \quad \left(\frac{x}{n}\right)^{\frac{x}{n}} = \left(\frac{1}{a^{\frac{1}{2B}}}\right)^{\frac{1}{n}}.$$

This last equation, by putting y for $\left(\frac{x}{n}\right)^{\frac{x}{n}}$, becomes

$$y^y = \left(\frac{1}{a^{\frac{1}{2B}}}\right)^{\frac{1}{n}}, \text{ or } y\downarrow, y = \frac{1}{n}\downarrow, \left(\frac{1}{a^{\frac{1}{2B}}}\right), \quad \text{(III.)}$$

A general solution of exponential equations of this form is given in "Dual Arithmetic a New Art," Part II., pp. 91 to 101.

In measuring heights by the barometer it is necessary to know, to the greatest nicety, the ratio of the density of mercury to that of air. The accurate and indefatigable M. Regnault found, at Paris, that a *litre* of air at 0° centigrade, under a pressure of 760 millimètres, weighed 1·293187 *grammes;* and at the level of the sea, in latitude 45°, it weighed 1·292697 grammes. He also found that a *litre* of mercury, at the temperature of 0° cent., weighed 13595·93 grammes. A *litre* of water, at its maximum density, weighs 1000 grammes; therefore the ratio of the density of mercury to that of air, in latitude 45°, will be=

$$\frac{13595{\cdot}93}{1{\cdot}292697} = 10517{\cdot}49.$$

At Paris, a *litre* of atmospheric air weighs 1·293187 grammes, but this number is only correct for the locality in which the experiments were made—that is, the latitude of 48° 50′ 14″, and a height of 60 mètres above the level of the sea. Taking 1·292697 grammes for the weight of a *litre* of air under the parallel of 45° latitude, and at the same distance from the centre of the earth as that at which the experiments were tried, to be 1·292697 grammes, then, putting w for the weight of the *litre* of air, in any other latitude, any other distance from the centre of the earth,

$$w=\frac{1{\cdot}292697\ (1{\cdot}00001885)\ (1-{\cdot}002837)\cos\lambda}{1+\frac{2q}{n}}. \qquad \text{(IV.)}$$

$n=6366198$ mètres, the mean radius of the earth, q the height of the place of observation above the mean radius, and λ the latitude of the place. In applying formula (IV.) to a particular example, the author found that at Philadelphia, U.S., lat. 39° 56′ 51″·5 N., the weight of a *litre* of air was 1·2914392 grammes; the ratio of the density of mercury to that of air at the level of the sea was 10527·735; and $n=6367653$ mètres at Philadelphia.

Regnault also found, by experiments, that 1·36706 represents the volume of air at 100° centigrade thermom., the volume at 0° being supposed = 1. Before the time of Regnault, these and many other constants were greatly in error. Experiments show that air, under the same pressure, expands uniformly for equal increments of temperature; that the expansion due to the same increase of temperature is not the same for all gases, as many scientific men have supposed. However, in air the expansion for a unit of bulk is ·36706, according to Regnault, from the freezing to the boiling points; and therefore the expansion for each degree of Fahrenheit is $\frac{1}{180}$ of ·36706.

Let α = the expansion of air for each degree of the thermometer, k = the ratio of the elasticity of air to its density at the temperature of melting ice; then the bulk at the temperature x will be increased, and therefore the density diminished in the ratio of $1+\alpha x$ to 1; consequently $k\,(1+\alpha x)$ = the ratio of the elasticity to the density at the temperature x.

$$\text{For the air let}\left\{\begin{array}{l}\left.\begin{array}{l}p=\text{the elasticity,}\\ g=\text{the force of gravity,}\end{array}\right\}\text{at the surface of the earth.}\\ \left.\begin{array}{l}P=\text{the elasticity,}\\ D=\text{the density at temp. }x,\\ G=\text{the force of gravity,}\end{array}\right\}\text{at the altitude }x\text{ above the surface;}\end{array}\right.$$

$$\text{Then }\frac{P}{D}=k\,(1+\alpha x),\ \text{and}\ \therefore\ P=D\,k\,(1+\alpha x).$$

Then, for what is conventionally termed the differential of P, in the notation of the differential calculus, put $d\,P$;

Then, $P-dP$ = the pressure at the altitude $(z+dz)$;
P = the pressure at the altitude z.

$$\therefore \quad -dP = \text{the difference of pressures} = D\,G\,dz = \frac{D\,g\,r^2\,dz}{(r+z)^2},$$

r being put for the radius of the earth where the observation is made.

$$\therefore \quad -\frac{dP}{P} = \frac{g\,r^2}{k\,(1+\alpha\,x)}\left(\frac{dz}{(r+z)^2}\right)$$

$$\therefore \quad \text{Intergration gives } \log_{\epsilon} P = \frac{g\,r^2}{k\,(1+\alpha\,x)}\left(\frac{1}{r+z}\right)+C,$$

$$\therefore \quad \log_{\epsilon} p = \frac{g\,r^2}{k\,(1+\alpha\,x)}\left(\frac{1}{r}\right)+C;$$

$$\text{Consequently, } \log_{\epsilon} \frac{p}{P} = \frac{g\,r^2}{k\,(1+\alpha\,x)}\left(\frac{1}{r} = \frac{1}{r+z}\right)$$

$$= \frac{g\,r}{k\,(1+\alpha\,x)}\left(\frac{z}{r+z}\right) = \frac{g\,z}{k\,(1+\alpha\,x)}\left(\frac{r}{r+z}\right); \qquad \text{(V.)}$$

From experiment it also appears that mercury contracts uniformly as its temperature decreases.

For mercury let $\left\{\begin{array}{l} H = \text{height of barometer,} \\ M = \text{density of mercury,} \\ T = \text{temperature of mercury} \end{array}\right\}$ at the surface of the earth,

and H_1, M_1, T_1 the same quantities respectively at the altitude z; and $\frac{1}{\beta}$ = the condensation of mercury for one degree of the thermometer;

$$\therefore \quad M_1 = M\left(1+\frac{T-T_1}{\beta}\right).$$

But $p = g\,H\,M$; and

$$P = G\,H_1\,M_1 = \frac{g\,r^2}{(r+z)^2}\,(H_1)\,M\left(1+\frac{T+T_1}{\beta}\right)$$

$$\therefore \quad \frac{p}{P} = \left(\frac{r+z}{r}\right)^2 \frac{H}{\left(1+\frac{T-T_1}{\beta}\right)H_1};$$

putting h for $\left(1+\frac{T+T_1}{\beta}\right)H$, and equating the hyperbolic logarithm of the last value of $\frac{p}{P}$ with the value before found (V.), we have

$$\frac{gz}{k(1+\alpha x)}\left(\frac{r}{r+z}\right)=\log._\epsilon\left\{\frac{H}{h}\left(1+\frac{z}{r}\right)^2\right\}$$

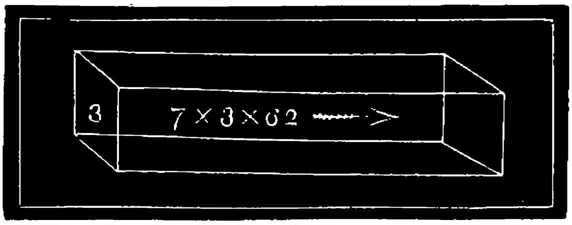
$$\therefore\quad z=\frac{k(1+\alpha x)}{g}\left(1+\frac{z}{r}\right)\log._\epsilon\left\{\frac{H}{h}\left(1+\frac{z}{r}\right)^2\right\}$$

The temperature x has been supposed to remain the same throughout the whole column z, whereas it always decreases as we ascend from the surface of the earth; but, being ignorant of the law of this change, a mean value (τ) between the values at the two stations is taken and considered constant; the mean τ being substituted for x in the last equation, (I.) is established.

In a discussion on General Anstruther's paper respecting the flight of projectiles, reported in the "Journal of the Royal United Service Institution," vol. ix., the author of the present work made some observations upon the value of the force of gravity and the constant g, the resistance of fluids to bodies moving in them, and on that due to friction when the moving body described an irregular path; those remarks and observations are introduced in this place for the student's most careful consideration. On the occasion referred to the author observed: When great minds conspire to perpetuate a fallacy, it has always been a difficult matter to clear that fallacy away. I know of no subject capable of being submitted to mathematical investigation that has received a greater amount of fallacious treatment, and that, too, by great minds, than the motion of projectiles. Initial velocities have been little more than guessed at, the resistance of the air overrated, and the force of gravity misstated. The system introduced by General Anstruther, which is practical and easily applied, must give correct results within the range of his experiments, without offering any special theory about initial velocities, the resistance of the air, or the force of gravity; indeed, in his system are collected all those elements. Let us examine the theories put forward with respect to the resistance of the atmosphere to the motion of a ball passing through it. The resistance of a fluid to the motion of a body is said to vary as the square of the velocity; the hypothesis upon which this law rests may be thus considered: the fluid displaced must have the same motion given to it as that of the moving body; hence, on this supposition, the units of work destroyed by the fluid will be equal to the accumulated work in the fluid.

Fig. 139.

Velocity=7 feet a second.



For example, let the cross section of the body perpendicular to the direction in which the body moves = 3 square feet, the weight of a cubic foot of the fluid = 62 lbs., and the velocity of the body 7 feet a second.

The weight of the fluid moved at a velocity of 7 feet a second = 7 × 3 × 62 lbs.

The units of work expended in the displacement will be

$$\frac{7^2\times 7\times 3\times 62}{2g}$$

in which g is put $=32\frac{1}{6}$, $32\frac{1}{3}$, 32, &c., and is said to vary inversely as the square of the distance from the centre of the earth.

Then the body in moving through 7 feet destroys

$$7\times\frac{7^2\times 3\times 62}{64},\text{ units of work.}$$

If T be the resistance of the fluid in pounds, the units of work destroyed will also be represented by T × 7;

$$\therefore\quad T\times 7=7\times\frac{7^2\times 3\times 62}{64},$$

$$\text{or } T=\frac{7^2\times 3\times 62}{64}.$$

Hence it is concluded that the resistance increases with the square of the velocity, as well as with the area of the cross section of the body presented to the fluid. It is easily perceived, especially with high velocities, that this law is not strictly true, and very far from the truth when the motion takes place in air. It has been found, from experiments on railways, that the resistance of the atmosphere to the motion of the train depends chiefly on the length of the train, and not upon the frontage of the carriages. The resistance resembles more that of friction than the moving of a long parallelopiped of the fluid in which the body moves.

Taking this latter view of the motion of a ball through the atmosphere from one given point to another, the units of work will always be the same, whatever may be the form of the curve described.

Let W, Fig. 140, be a body moved through the air, then, by the resolution of pressures, the pressure of W, perpendicular to the tangent $= W \cos \alpha$;

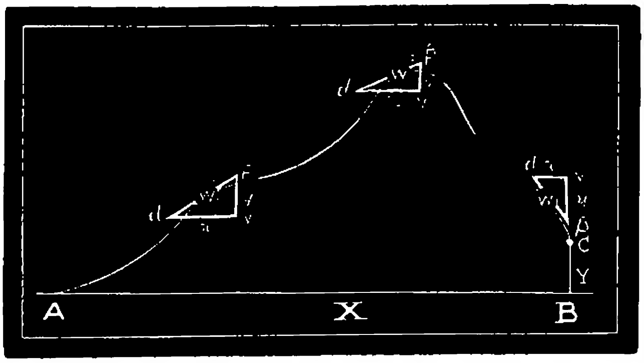
$$\therefore \text{ The resistance of friction} = f W \cos \alpha,$$

putting f for the coefficient of the friction of the air on the ball. Hence the total work on the plane $\alpha\beta = z f W \cos \alpha + W y$; but $z \cos \alpha = x$;

$$\therefore \text{ The work on each plane} = f W x + W y,$$

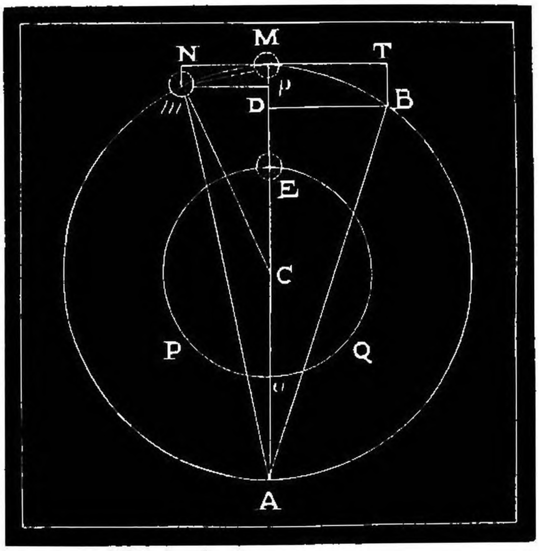
this expression being independent of the inclination of the plane. It points out the work employed to move the body over the horizontal distance $\alpha\gamma$, added to the work due to the gravity in elevating the body the vertical space $\beta\gamma$. Since a curve may be supposed made up of an infinite number of straight lines, therefore the work upon the whole curve will be equal to the work done upon the horizontal projection A B, added to the work done in opposition to gravity in raising the body from B to C, no matter what form be given to the curve A W W W C. This demonstration shows that the way in which General Anstruther has treated the resistance of the air to be the most rational yet proposed. In the next place, let us examine the manner in which the value of g, representing the force of gravity, has been estimated.

Fig. 140.


d
W
F
v
d
W
F
v
d
W
A
C
Y
A
X
B

Let r=C M=radius, Fig. 141; v=the uniform velocity of M on the curve; *t*=the time employed in moving from M to *m*, or in falling from M to *p*. If *f* be considered constant during a short space of time,

Fig. 141.


N
M
T
p
D
B
m
E
C
P
Q
a
A

$$f\,dt = dv$$

$$\therefore \quad f t = v = \frac{ds}{dt}$$

$$\therefore \quad s = \frac{f t^2}{2} = Nm = Mp.$$

m M = v t = uniform velocity × time.

Again, suppose the time so short that the arc is equal to the chord, or arc m M = chord m M.

Then $\mathrm{Mp} \times \mathrm{MA} = (\text{chord Mm})^2 = (\text{arc Mm})^2 = (\mathrm{v\,t})^2$;

$$\text{but, } \mathrm{Mp} = \frac{f\mathrm{t}^2}{2} \text{ and } \mathrm{MA} = 2\mathrm{r},$$

$$\therefore \quad \mathrm{Mp} \times \mathrm{MA} = \frac{f\mathrm{t}^2}{2} \times 2\mathrm{r} = \mathrm{v}^2\mathrm{t}^2$$

$$\therefore \quad f = \frac{\mathrm{v}^2}{\mathrm{r}}; \quad \text{(A)}.$$

If T be the time of an entire revolution, then

$$\mathrm{Tv} = 2\pi\mathrm{r} \qquad \therefore \quad \mathrm{v}^2 = \frac{4\pi^2\mathrm{r}^2}{\mathrm{T}^2}$$

$$\text{and } f = \frac{4\pi^2\mathrm{r}}{\mathrm{T}^2}; \quad \text{(B)}.$$

Let another body, E, Fig. 141, describe the circumference of a circle, E P Q, in the time T_1; and let F represent the accelerating force of gravity at E. Put R = E C, then, from (B)

$$\mathrm{F} = \frac{4\pi^2\mathrm{R}}{\mathrm{T}_1^2}$$

$$\therefore \quad f : \mathrm{F} :: \frac{\mathrm{r}}{\mathrm{T}^2} : \frac{\mathrm{R}}{\mathrm{T}_1^2} \qquad \text{(C)}.$$

Now the third law established by the experiments of John Kepler shows that the squares of the times are as the cubes of the distances from the centre, C; hence if we put r^3 for T^2, and R^3 for T_1^2, then C becomes

$$f : \mathrm{F} :: \frac{1}{\mathrm{r}^2} : \frac{1}{\mathrm{R}^2} \quad \text{(D)};$$

and thus it is shown, by the experiments of Kepler, that the force of gravity varies inversely as the squares of the distances. Sir Isaac Newton had little to discover in this matter when Kepler's laws are admitted. The experiments at General Anstruther's command are as accurate as those upon which the law indicated by (D) is founded. The influence of gravity on the motion of the moon round the earth, or of the earth round the sun, is then compared with the falling of a heavy body near the earth's surface, and near the earth's surface f is represented by g, which is said to be found by experiments. Astronomers apply a variety of corrections to uphold (D), and the cooked-up value they have given to g. When g is trimmed up to suit the motions of the heavenly bodies, it is out of gear to apply it to the falling of heavy bodies in air near the earth's surface. In changing the value of g, General Anstruther is right, for his system corrects the constant quantities, and secures correct results within the range of his limiting experiments.

To find whether the resistance of the air or any other fluid medium is proportional to the square of the velocity (V) or not, and also to find whether the value usually given to (g), the force of gravity near the surface of the earth is under or over estimated,

Generally g is put$=31\frac{1}{6}$, $32\frac{1}{3}$, 32, &c.

Let v^2 multiplied by some constant coefficient express the retarding force; and to simplify the investigation put this coefficient under the form n^2g.

Then the retarding force will be expressed by

$$\mathrm{g}-\mathrm{g}n^2v^2; \quad \text{(I.)}.$$

In this investigation s is put for the space passed over in the time t.

In general terms, if F be any accelerating force, U the velocity, S the space, and T the time, the relations between these quantities are found from the two equations

$$\mathrm{U}=\frac{\mathrm{dS}}{\mathrm{dT}} \text{ and } \mathrm{F}=\frac{\mathrm{dU}}{\mathrm{dT}}; \quad \text{(II.)}.$$

When T is eliminated from these equations, then

$$\mathrm{UdU}=\mathrm{FdS}; \quad \text{(III.)}.$$

Again eliminate U, then we have

$$\mathrm{FdT}=\frac{\mathrm{d}^2\mathrm{S}}{\mathrm{dT}}; \quad \text{(IV.)}.$$

Then from (I.) substitute $g(1-n^2v^2)$ for F in (II.), then,

$$gdt=\frac{dv}{1-n^2v^2}; \quad (V.).$$

$$\text{From (III.), } gds=\frac{vdv}{1-n^2v^2}; \quad (VI.).$$

When the equations (V.) and (VI.) are integrated and corrected, on the supposition that when $t=o$; $v=o$ and $s=o$; the resulting equations will be

$$v=\frac{1}{n}\frac{\epsilon^{2gnt}-1}{\epsilon^{2gnt}+1} \text{ and } s=\frac{1}{2gn^2}\log.\frac{1}{1-n^2v^2}.$$

Eliminate v and equation (VII.) is obtained

$$s=\frac{1}{n^2g}\log.\frac{1}{2}\left\{\epsilon^{ngt}+\frac{1}{\epsilon^{ngt}}\right\} \quad (VII.).$$

The reduction may be effected as follows:

$$v^2=\frac{1}{n^2}\frac{(\epsilon^{2ngt}-1)^2}{(\epsilon^{2ngt}+1)^2} \quad \therefore \quad n^2v^2=\frac{(\epsilon^{ngt}-1)^2}{(\epsilon^{ngt}+1)^2}$$

$$1-n^2v^2=\frac{4\epsilon^{2ngt}}{(\epsilon^{2ngt}+1)^2}$$

$$\text{Then } \frac{1}{1-n^2v^2}=\frac{(\epsilon^{2ngt}+1)^2}{4\epsilon^{2ngt}}=\left(\frac{1}{2}\right)^2\left(\frac{\epsilon^{2ngt}+1}{\epsilon^{ngt}}\right)^2$$

$$=\left\{\frac{1}{2}\frac{\epsilon^{2ngt}+1}{\epsilon^{ngt}}\right\}^2$$

$$=\left\{\frac{1}{2}\left(\epsilon^{ngt}+\frac{1}{\epsilon^{ngt}}\right)\right\}^2$$

$$\therefore \quad s=\frac{1}{2n^2g}\log.\frac{1}{1-n^2v^2}=\frac{1}{n^2g}\log.\frac{1}{2}\left(\epsilon^{ngt}+\frac{1}{\epsilon^{ngt}}\right).$$

(VII.) may be put under the form

$$2\epsilon^{sgn^2}=\epsilon^{gnt}+\frac{1}{\epsilon^{ngt}} \quad (VIII.).$$

Now from (VIII.) the values of n and g may be found by dropping cannon-shot from a balloon, at different known heights, the time occupied in falling being measured. Two different values of s and two corresponding values of t being thus determined, (VIII.) presents two independent equations, from which n and g may be eliminated by the dual method of solving equations. I have but little doubt that the liberty taken by the General to change the reputed force of gravity near the surface of the earth, to meet the practical necessity, will be warranted when the value of g and the resistance of the atmosphere are independently and accurately determined. The force of gravity is pulled in all directions by astronomers, to make it fit the motions of the heavenly bodies; yet, after all the corrections and allowances are made, the moon and asteroids will not obey. The practical results given by the system of General Anstruther also show that it is right to change the initial velocities previously conjectured, for it is easily demonstrated that the initial velocities given, both by the ballistic pendulum and the pendulum of Navez, are erroneous. These instruments may be employed to indicate any conjectured initial velocity, for the time incorporated in the calculation is made to depend on a body very much restrained, falling in air a few inches. To say the least of the process, it is very ridiculous; the results may be corrected until the time anticipated is obtained. The value of g in or out of air is unknown, and yet it is made to tell all about the pendulum, and in return the pendulum is made to tell all about g. The General points out the errors of other systems in bold relief, and arrives at practical results within the range of his guiding experiments, without assuming initial velocity, a particular value for g, or including any wild theory about the wonderful resistance of the atmosphere to cover short-comings.

With respect to the force of *gravity*, and G, g, g, see pp. 37, 38, 93, 97, 134. The weight of a body is in a ratio compounded of its *mass* and the intensity of gravity in the place where it is situated, see p. 134. Hence, if we denote this weight by W, the mass by M, and the measure of *gravity* by g, we have

$$W=gM.$$

This quantity g, which is independent of the particular nature of the body, is in this position, the weight of what has been arbitrarily assumed to represent the unit of mass.

We may also write,

$$W = wV,$$

w denoting the weight of the body, which represents the unit of volume, and V its volume. The weight, w, is what is termed the specific gravity of the body that is considered, a denomination which is evidently improper, for gravity is common to all bodies of different species, and therefore there ought to be substituted for it the denomination *specific weight.*

Finally, if D represents the *mass*, under the unit of volume of the body which is considered, D will be what is termed the *density* of the body, and we shall have

$$M = DV,$$

$$\text{and since } W = gM, \quad \therefore \; W = gDV.$$

The preceding are the equations which obtain between the five quantities W, g, M, D, V, each of which should be expressed numerically, by referring it to a unit of its species. The student should bear in mind that the exact size and form of the earth has not been accurately determined.

Morin and other mechanical investigators first establish theories on abstract reasoning, and afterwards correct them by experiments; while others, like Anstruther, without any particular hypothesis, interpolate a range of experiments by proportion or empirical rules. A third class, which is very numerous, assumes constructions, and asserts that those constructions possess particular properties, that may or may not exist, and supports such claims by high-sounding pretensions, and a little mathematics to cover the delusion; a fair representative of this class is the expounder of Koch's ingenious system of skeleton structures.

CHAPTER IX.

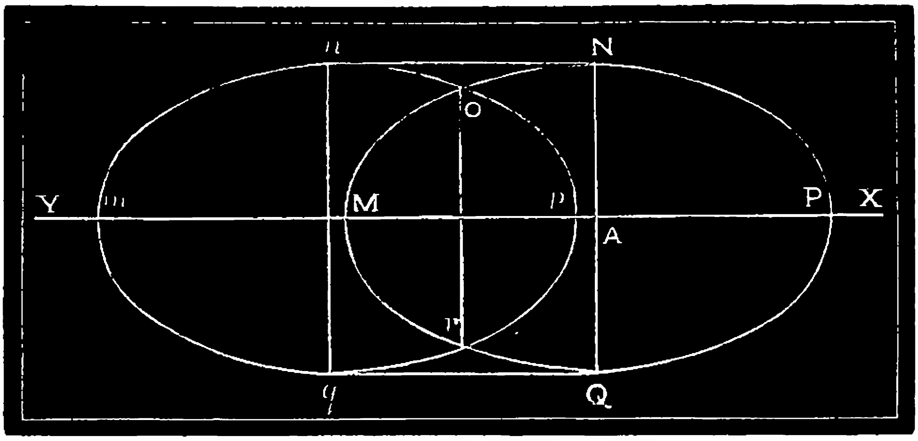
RESISTANCE OF FLUIDS, RESISTANCE OF WATER TO THE MOTION OF PADDLE-WHEELS, SCREW PROPELLERS, &c. &c.

THEORETICAL OBSERVATIONS AND CONSIDERATIONS.

When a body moves in a fluid, it necessarily displaces the particles of the fluid, impressing on them velocities in a certain ratio to its own, and it is readily conceived that the inertia of the fluid particles, thus set in motion, develops a resistance which increases with the velocity of the body. Similar effects are produced when a body is at rest or in motion, and acted upon by a fluid. The manner in which the fluid particles are divided when coming in contact with the body depends much upon the form of the body, and we perceive, without experiment, that the resistance must vary considerably with these circumstances.

When a body, Fig. 142, of any form, P Q M N, moves in a stream in a direction X Y, if we suppose each particle of the body transferred to a plane perpendicular to the direction of the motion, it is readily observed that in moving an elementary space $s = \text{M}\,m$, the body will displace a volume of liquid that may be represented by A s, found by multiplying the projection of the body upon the plane perpendicular to the direction X Y, by the length of the path $= s$.

Fig. 142.



In the two successive positions the body occupies the same volume, and there is a space common to the two positions, which corresponds to M O *p r*, so that the anterior volume, *o n m q r* M, is evidently equal to the volume O N P Q *r p*. It is further evident that each of these volumes is also equal to the volume, N *n q* Q, made by the greatest cross section of the body, or by the area of its projection, which we have represented by A.

Thus, when the body describes in relation to the fluid, or when the fluid describes in relation to the body, an elementary space *s*, the volume of the deviating fluid which passes from the front to the rear of the body is expressed by A *s* which we shall put $= q$; the *mass* will therefore be $=$

$$q\frac{d}{g} = d\frac{\mathrm{A}s}{g},$$

putting *d* for the density or weight of a cubic foot of the fluid. This deviating *mass* effects its relative displacement with a velocity depending essentially upon that of the body in relation to the fluid, in the case when it is the body that moves, and which it is reasonable to suppose is proportional to the velocity of the body. It will be the same for the units of work, or for the *vis viva* imparted to the deviating fluid; so that in the case of a fluid at rest, in which moves a body impressed with a velocity V, the *vis viva* imparted to the displaced fluid for an elementary motion of the body will be proportional to

$$\frac{d\,\mathrm{A}\,s}{g}\mathrm{V}^2,$$

and if we put k=the unknown ratio of the *vis viva* F, really impressed upon the fluid, to the above expression, we shall have

$$\mathrm{F}=k\frac{d\,\mathrm{A}\,s}{g}\mathrm{V}^2;$$

in which k has to be found by experiment, and

$$\frac{\mathrm{F}}{2}=\text{the units of work.}$$

On the other hand, if we put R=the total resistance which the inertia of the fluid particles opposes to their displacement, *the work* of this resistance for the elementary space, s, will be

$$\mathrm{R}\,s,$$

and must, according to the general principle of *vis viva* (see pp. 116, 117, 118), be equal to one-half the *vis viva* imparted to the displaced fluid.

$$\therefore\quad \mathrm{R}s=\frac{1}{2}\,k\,\frac{d\mathrm{A}s}{g}\mathrm{V}^2,$$

$$\therefore\quad \mathrm{R}=\frac{kd\mathrm{A}}{2g}\,\mathrm{V}^2.$$

In the case where the fluid displaced by the body is moving with a velocity of its own, if the body moves in an opposite direction to the motion of the fluid, the relative velocity with which the fluid particles are met and displaced is $\mathrm{V}+v$; and when the two velocities V and v are in the same direction, is $\mathrm{V}-v$; similar reasoning to that already employed will give us, then, for the case in which the body moves in an opposite direction to the fluid

$$\mathrm{R}=k\,d\,\mathrm{A}\frac{(\mathrm{V}+v)^2}{2g},$$

and for that in which they move in the same direction

$$R = k\,d\,A\frac{(V-v)^2}{2g}.$$

THEORETICAL EXPRESSION FOR THE UNITS OF WORK DEVELOPED EACH SECOND BY THE RESISTANCE OF WATER, AIR, OR ANY OTHER FLUID MEDIUM.

When all the circumstances of motion remain the same, and the phenomena occur continuously in the same manner, *the work* developed in each second by the resistance of the medium opposed to the motion of the body is, in the case of a fluid at rest,

$$\mathbf{R\,V} = k\,d\,\mathbf{A}\,\frac{\mathbf{V}^3}{\mathbf{2}g},$$

and *the work* each second in the case of the fluid being in motion is

$$\mathbf{R\,V} = k\,d\,\mathbf{A}\,\frac{(\mathbf{V} \pm v)^2}{\mathbf{2}g}\,\mathbf{V},$$

which shows that, in the first case, *the work* of resistance increases as the cube of the velocity.

EQUIVALENT EXPRESSIONS FOR THE RESISTANCE OF FLUIDS TO MOVING BODIES.

In the preceding expression of the resistance applied to a liquid whose density, d, is constant, and in places where the value of $2g = 64{\cdot}3634$, and putting K for $\frac{kd}{2g}$ it then takes the form

$$R = K\,A\,V^2,$$

$$\text{or}\quad R = K\,A\,(V \pm v)^2,$$

in which forms the expression for the resistance is frequently used by writers on mechanics.

Some authors, and in particular Dubuat, call H the height due to the relative velocity V or $V \pm v$, and consequently

$$H = \frac{V^2}{2g}, \text{ or } H = \frac{(V \pm v)^2}{2g},$$

and putting $K^1 = Kd$, write this formula under the form

$$R = K^1 A H.$$

It is evident that these three expressions for the resistance, R, are equivalent; the last two are pointed out so that the student may understand the authors who employ them. These latter expressions give but a poor idea of the law of resistance.

EXPERIMENTS UPON THE RESISTANCE OF WATER TO THE MOTION OF BODIES WHICH WERE DIFFERENTLY FORMED. BY GENERAL ARTHUR MORIN. METZ, 1836 AND 1837.

The bodies subjected to these experiments were :—

1st. Thin plates of different sizes, which were made to ascend from the bottom to the surface of the water by the action of a counter-weight.

2nd. Solid or hollow brass spheres, with diameters ranging from 0·341 ft., 0·387 ft., 0·422 ft., 0·485 ft., to 0·530 ft.

3rd. Tin-plate cylinders, with altitudes equal to their diameters, which were 0·324 ft., 0·656 ft., 0·984 ft.

4th. Cones terminating upon cylinders, with the same diameter and height as the preceding, and whose angles at the summit varied as follows:

Half-angle at the summit, 64° 48′; 46° 50′; 26° 01′; 18° 49′; 14° 19′ 48″.

5th. Cylinders of the same dimensions with the preceding, and terminated in front by hemispheres.

Mode of Observation.—The experiments were made upon the Moselle, in front of the dam at Pucelles, in a place where the water was, at least at its surface, nearly without a current, and had a depth of 16·4 ft. It was the most suitable place that could be found in the neighbourhood.

The vertical motion of the body was made in the descent, by its own weight, with an occasional ballast to increase its velocity, and in the ascent by means of a counter-weight. In all cases, the law of the motion was observed and determined by means of a chronometric apparatus with a style, similar to those used in the experiments on friction.

In the first experiments it was at once apparent that the resistance of the water increased so rapidly with the velocity, that the motion very readily became uniform. Then, knowing, in each case, the velocity and the motive weight, and keeping account of the passive resistances, it was easy to calculate the value of the corresponding resistance of the fluid, and to investigate its law.

A graphic representation of the results, taking the resistances for abscissæ, and the squares of the velocities for ordinates, has shown in each case, as in the preceding, that the resistance is composed of two terms, the one independent of the velocity and simply proportional to the wetted surface, the other proportional to the square of the velocity; but here the first term is so small, that it may be neglected in relation to the second, as soon as the velocity has become 3·28 ft. per second.

According to this, the resistance opposed by the water to bodies, as proved by these experiments, would be represented by the formula,

$$R = K A V^2,$$

A being the projection of a body upon a plane perpendicular to the direction of motion.

The values of K, derived from experiment, are entered in the following table:

Values of the coefficient K *of the formula* $R=KAV^2$.

Bodies used.				Values of K.
Thin plates (rising vertically upwards)..				2·724
Spheres..				0·41969
Right cylinders with height equal to diameter				1·7715
Cylinders with same proportions, terminated by right cones, whose heights are to the radii of their bases in the ratio of.................	0·94 to 1	Corresponding angles at the summit.	64° 48′	1·3944
	1·89 to 1		46° 50′	1·0276
	4·05 to 1		26° 1′	0·90868
	5·92 to 1		18° 49′	0·84302
	7·66 to 1		14° 19′ 48″	0·77449
Cylinders of the same proportion terminated by spheres				0·77487

Observations upon these results.—The value of the coefficient K found in the experiments for thin plates is considerable, and nearly double that found by Dubuat in causing a vertical plane to move in a horizontal direction, thus producing a displacement of water entirely different from that in our experiments, and occasioning the difference of results.

It is remarkable that of all the bodies used, the spheres offered the least resistance, and that cylinders, terminated by half spheres, have experienced less than those with acute cones.

This result shows, that in regard to the resistance of a medium, the spherical form for projectiles, and the semicircular for piers of bridges, are the most favourable.

Influence of the acuteness of the angles of cones upon the resistance.—In comparing the values of the half angles at the summit of the cones, expressed in fractions of the semi-circumference, with the values of the resistance, we see that the coefficient K of this resistance increases proportionally with these angles; starting from a certain value answering to the angle zero. It may be given by the formula

$$K=0{\cdot}59005+2{\cdot}2998a,$$

a being the half of the angle at the summit in terms of a fraction of the semi-circumference. The comparison of the values of K given by this formula, with those deduced directly from the experiment, is established in the following table:

Comparison between the values of the coefficient K *as deduced by formulas and by experiments.*

Half angles at the summit in fractions of the semi-circumference.	Values of the coefficient K deduced	
	From the formula.	From experiment.
0·500	1·739	1·771
0·362	1·422	1·394
0·262	1·192	1·027
0·145	0·923	0·908
0·105	0·831	0·843
0·080	0·774	0·774

We see that, with the exception of the case relative to the cone whose half-angle at the summit was measured by an arc equal to 0·262 of the semi-circumference, the results, including even that pertaining to the plane base of the cylinder, are quite correctly represented by the formula, and that we may use it for intermediate cases, which have not been experimented upon.

Experiments upon the resistance of water to the motion of projectiles.—Without entering into details, for which this is not the proper place, I must say something about the remarkable results of experiments made by me in common with MM. Piobert and Didion, at Metz, in 1836, upon the penetration of projectiles in water.

These experiments were made at the basin which had served for the beautiful hydraulic researches of MM. Poncelot and Lesbros, in firing horizontally beneath the surface of the water projectiles which penetrate the water after having traversed an orifice formed by spruce scantling. A horizontal flooring, placed at the bottom of the basin, and marked with strips, received the projectiles, which always reached it with a very small velocity.

We found, with this arrangement, the resistance offered to solid balls with diameters of 0·354 ft., 0·328 ft., 0·530 ft., and 0·72 ft., and to shells of the same diameters, having different thicknesses and weights, the initial velocities of the projectiles varying from 229 ft. to 1640 ft. in 1″.

From the general view of all the experiments made, the results of which are published in No. VII. of the "Memorial de l'Artillerie," we conclude that the resistance of water to the motion of these projectiles may be represented by the formula

$$R=0{\cdot}453\ AV^2 \text{ lbs.},$$

while the experiments above cited (No. 305) gave us

$$R=0{\cdot}4197\ AV^2 \text{ lbs.}$$

On the other hand, the ancient experiments due to Newton, and made by observing the time of the fall of spheres in water, lead to the value

$$R=0{\cdot}46498\ AV^2 \text{ lbs.},$$

and those which Dubuat made in causing spheres placed at the end of the arm of a horse-gin to pass in a circle through the water, furnish the formula

$$R=0{\cdot}4197\ AV^2 \text{ lbs.}$$

An inspection of all these researches, made by processes so different, and within limits so extended, enables us to conclude that, in liquids, the law of the proportionality of the resistance to the square of the velocity, is applicable to spheres, even when moving with the highest velocities.

The preceding theoretical formulæ apply to boats which navigate the sea, rivers, and canals; but their results are influenced by different circumstances, of which it is important to take an account; some are permanent, others accidental.

ANALYSIS OF MORIN'S EXPERIMENTS ON THE RESISTANCE OF WATER TO THE MOTION OF PADDLE-WHEELS.

In the first set of experiments two models were employed, the *first wheel* was 3·31 ft. diameter, and received the paddles, which were varied in number up to twenty at most. The dimensions of the paddles employed on this wheel were as follows :

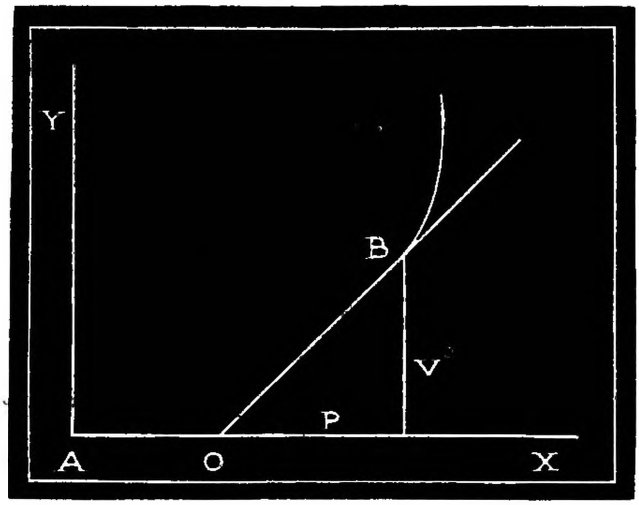
In length parallel to the axis, ·328 ft. ; ·656 ft. ; ·984 ft. ; 1·968 ft.
Breadth, in the direction of the radius ·328 ft. ; ·659 ft. ; 1·148 ft. ; ·659 ft.

The shaft of the wheel formed a windlass around which rolled a cord, which passed to the summit of a crane 55·77 ft. in height, which supported a box in which was placed the motive weight. The wheel was established upon a fixed frame, and the depths of immersion were varied at will, in raising or lowering the level of the water. The velocity of rotation of the wheel varied from the lowest in which it was possible to observe a regular motion up to 19·68 ft. a second. When the motions became uniform, the velocities were observed by a timepiece indicating tenths of seconds.

The *second wheel* employed was 8·567 ft. in exterior diameter, with paddles 2·29 ft. long in the direction of the axis, by a breadth of 1·659 ft. in the direction of the radius in which they were placed. The depth of immersion of these paddles was successively 1·659 ft., 1·325 ft., and ·937 ft. For each number of paddles, and each depth of immersion, the motive weights, and consequently the velocities, were gradually changed, so as to have a series of experiments in which one element only was variable.

Having thus, for each case, the values of the resistance corresponding with the different velocities, a graphic representation, Fig. 143, was made of all the results in taking for abscissæ, along AX, the motive weights, and for ordinates, in the direction of AY, the squares of the velocities of the middle point of the immersed section. In all the series so represented it was observed, that up to a certain velocity, all the points were on a straight line OB, which cut the line of abscissæ AX in front of the origin

Fig. 143.



at a point O, variable for each curve, which shows that the abscissæ or the resistance was in each case represented as for boats moved by horses, by an expression of the form

$$R=K_2A+K_1AV^2,$$

V=the velocity of the middle submerged section of the paddle; K_1 and K_2 constant coefficients.

The immersed surface A of the paddles was determined from the number of paddles simultaneously submerged, in whole or in part, by calculating the sum of the immersed portions of the floats for many successive positions of the wheel, and taking the mean of the sums of the areas thus found. A thus represents the mean value of the total resistance of the paddles acting upon the water.

The method of representation, Fig. 143, gives the value of the constant K_2; for the abscissæ OA of the point O of the straight line expressing the law of resistance, was that of the term K_2A. The following values were found in this manner:

Dimension of paddles. 20 in number.	Total submerged area.	Constant resistance.	
		Derived from the trace K_2A.	A square foot K_2.
Feet.	Square feet.	lbs.	
·65 by ·65	1·4693	·2867	·19513
·98 by 1·15	4·677	·88219	·18862
·97 by ·65	4·4241	·86013	·19442
Mean . . .	. . .	. . .	·19272

The construction, Fig. 143, also enables us to supply the value of the coefficient K_1 of the term assumed to be proportional to the square of the velocity, since the inclination of the straight line expressing the law of resistance is given by the expression

$$K_1 A = \frac{R - K_2 A}{V^2}$$

$R - K_2A$ being the value of the abscissæ of this straight line OB, diminished by OA; and V^2 being the values of the ordinates. Dividing in each case the values K_1A, given by experiment, by the known surface A, we thus find the values of the coefficient K_1.

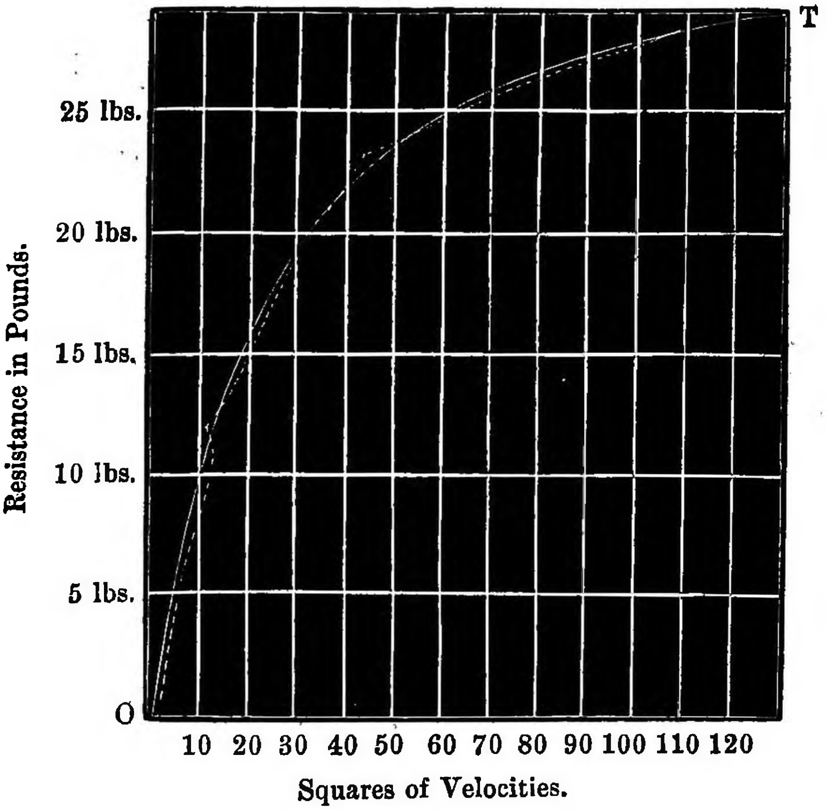
Causes which alter the law of resistance assumed to be expressed under the form

$$K_2A + K_1AV^2.$$

The student must not forget that Morin assumed the above form.

Before giving the values of the coefficient K_1, furnished by the summary of the experiments, it is necessary that we should point out a circumstance, which, in altering the conditions of the phenomena, exerts a considerable influence upon the results. In order that with different velocities and depths of immersion, the wheel and its floats may be in comparable conditions, it is necessary, as we have hitherto implicitly admitted, that the void formed by the paddles, which have driven before them the water upon which they have acted, should be constantly replaced, so that the next paddle submerged may meet the same resistance. Now, in observing the motion of the return of the water into the void, we readily understand that the refilling must be accomplished by the flowing of the surface, as it were, over a dam at the sides, and that a certain time is required for its operation. If, then, the wheel turns so rapidly that the void has no time to fill, the paddles no longer finding the same quantity of water to drive as in less velocities, the circumstances of the phenomena are changed, and accordingly the law of resistance must be modified. This change increasing more and more with the velocity, it happens that the paddles meet a less amount of water, which may at last appear as if the wheel turned in air instead of water. All these effects are represented by the trace giving the results of the experiments, Fig. 144, relating to a series of experiments made with 20 paddles submerged ·344 ft.

Fig. 144.



The curved line OT, Fig. 144, representing the law of resistance at first near O, is almost straight; this straightness is prolonged up to a certain velocity, depending upon the depth of the immersion and the distance apart of the paddles; but beyond this velocity it departs more and more, showing that the resistance no longer maintains its proportion with the square of the velocity. All these facts are highly important to steam navigation, for they show that it is necessary to establish between the depths of immersion of the paddles, their distance apart, and the velocity with which they are impelled, such relations that the water may always have time to fill up the voids, and that for each wheel, screw, or paddle, there is a limit of speed adopted to the best effect.

To find the value of K_1 *of the second term of the expression for the law of resistance.*

Having regard to the circumstances pointed out, and consequently restricting the law of the resistance within the limits of his ability to verify them, Morin makes known the results relating to experiments in which the velocity of the wheel and the spaces of the paddles allowed a complete return of the water.

The values of the coefficient K_1, furnished by the experiments, were as follows:

Values of the coefficient K_1 *of the formula*

$$R = A\,(K_2 + K_1 V^2).$$

	Number and dimensions of the paddles.		Values of K_1.
Wheel 3·31 ft. in diameter.	10 paddles 0·33 ft. by 0·33 ft.		1·990
	5 ” ” ” ”		2·0748
	10 ” 0·66 ft. ” 0·66 ft.		1·9319
	5 ” ” ” ”		2·0794
	5 ” 0·98 ft. ” 1·15 ft.		2·2365
	10 ” 1·97 ft. ” 0·66 ft.		1·9928
	5 ” ” ” ”		2·2460
Great wheel 8·567 ft. in diameter.	8 paddles of 2·29 ft. submerged...	1·659	2·2269
		1·325	2·1698
		0·937	2·4078
	General mean.........		2·1355

The general mean does not differ over $\frac{1}{11}$ from the partial results, and we see that when the spaces of the paddles were within the indicated limits, that the effect exerted by wheels with plane floats upon the axle may be represented by the formula

$$R = A\,[\cdot 19272 + 2\cdot 13559 V^2],$$

A being the mean of the surfaces of paddles simultaneously submerged at rest, V the absolute velocity of the wheel.

Case where the wheel turned in running water.—The wheel which had served for the above experiments having been placed in a small wooden canal, 3·7 ft. wide by 2 ft. deep, the same course was taken to ascertain the law of resistance. Without going into unnecessary details, Morin states that the results of these new experiments were also represented, with sufficient exactness for practice by the same formula, by adding or subtracting the velocity v of the current to or from that of V, so that the general expression of effort exerted upon still or running water by the paddles of wheels with plane floats will be

$$R = A\,[0\cdot 1927 + 2\cdot 0785\,(V \pm v)^2],$$

in taking for the coefficient of the second term a number which conforms best to all the experiments.

Influence of the presence of a boat near the wheels.—The experiments in question were made upon isolated wheels, and it was proper to ascertain whether the presence of a boat near the wheel would exert any influence upon the intensity and law of the resistance.

For this purpose, the experimenter placed near the wheel, at a distance of 0·13 ft., parallel to the exterior vertical plane of the floats of the wheel, 8·567 ft. in diameter, a boat submerged an equal depth with the floats, and made two sets of experiments, with depths of immersion of 1·325 ft. and 0·84 ft., to compare the results with those of the series made in the case when the wheel was isolated, and its paddles immersed 1·325 ft. and 0·937 ft.

The results of these experiments seem to show that by reason of the obstacle which the presence of the boat opposed to the return of water into the void formed by the float, the resistance diminished somewhat, but so small a quantity that it may come within the limits of the errors of observation. In fact, we found

At the depth of immersion of 1·325 ft. without boat $K_1=2{\cdot}1698$
” ” ” ” with boat $K_1=2{\cdot}1413$
At the depth of immersion of 0·937 ft. without boat $K_1=2{\cdot}4078$
” ” ” 0·84 ft. with boat $K_1=2{\cdot}1580$

We see, then, that the preceding formula derived from a summary of the experiments may be still applied to the case where the wheel is placed at the side of a steamboat.

Application to the wheels of steamboats.—The formula of the resistance experienced, and the effect transmitted by the paddles of a wheel turning in water being

$$R=K_1AU^2=2{\cdot}13559AU^2,$$

when the axis of this wheel has no motion of translation, it is clear that if this axis is borne upon a steamboat going with a velocity V, the paddles will only impinge upon the water with a velocity $U-V$, and that in this case the formula expressing the resistance experienced by the floats will be

$$R=K_1A(U-V)^2$$

in still water, and, finally, that if the boat bearing the wheel ascends or descends a stream running with a velocity v, the expression of resistance will be

$$R=K_1A(U-V-v)^2 \text{ on the ascent,}$$
$$R=K_1A(U-V+v)^2 \text{ on the descent.}$$

If we examine particularly the case of navigation in still water, the work of this resistance, or that of the machine moving the wheel in 1″, will be

$$RU=K_1A(U-V)^2U \text{ units of work,}$$

and if we express in horse powers of 550 units of work a second, the effective force of the motor will be

$$\frac{R\cdot U}{550}=N=\frac{K_1A(U-V)^2}{550}.$$

An observation of existing constructions will allow us to judge whether the value of the coefficient K_1 derived from the experiments above reported, agrees with the observed facts of navigation. Indeed, we have for each boat the dimensions of the wheels and floats, and the number of the latter, from which we can deduce the submerged surface of the paddles.

Observation gives us the velocity, U, of the paddles, which, by reason of their small height compared to the radius of the wheels, may be regarded as the point of application of the resistance, as well as the velocity, V, of the boat; and if we introduce the value of $K_1=2{\cdot}13559$, derived from our experiments, the above formula should give the effective force of the machine, such as observation has furnished. Direct experiments, made by hauling upon a fixed point, in giving the effort exerted and transmitted by the paddles to set the boat in motion, enable us to verify directly the formula

$$R=2{\cdot}13559\ AU^2,$$

by introducing the particular data of each case.

In making this comparison upon the steamers the *Sphinx*, the *Mentor* of 160 nominal horse power, the *Médée*, and the *Véloce* of 220 horse power, for which the dimensions and different velocities are given by M. Campaignac, in his work upon steam navigation, we have the following data and results:

Names of steamers.	Horse power of each of the two engines. N.		Area of paddles submerged for each wheel. A.	Velocity of the middle of the paddles. U.	Velocity of the boat. V.	Value of the coefficient K_1.
	nominal.	effective	sq. ft.	ft.	ft.	
Sphinx	80	80·851	41·227	19·993	15·190	2·3074
Mentor	80	80·854	37·136	20·862	15·528	1·9897
Médée	110	110·380	56·371	20·823	16·201	2·4318
Véloce	110	111·350	42·928	20·948	15·948	2·2879
						2·3542

It should be observed that the value of the whole simultaneously submerged area of the paddles was determined by tracings, and on the supposition of the vertical floats being entirely submerged a little below the surface, but probably less than it was in reality; so that the values of K_1 are undoubtedly greater than they should be. It is not, then, surprising that the mean of these values surpasses those derived from the experiments superintended by General Morin.

PERFORMANCE OF PADDLE-WHEEL STEAMERS.

Let a = area of all the floats immersed, in square feet;
b = the immersed angle of the paddle-wheels, at the centre of pressure;
c = radius from the centre of motion to the centre of pressure of the floats;
d = acting area in square feet.

$d = (a)$ multiplied by the natural cosine of $\frac{b}{3}$.

Ex.—Suppose the area of all the immersed paddles, a= 312 square feet, the angle $b=48°$; what is the acting area?

$$\frac{48°}{3}=16° ; \quad \text{Natural cosine} = \cdot 9612617,$$

$$312 \times \cdot 9612617 = 299 \cdot 9136504 \text{ square feet,}$$

300 square feet acting area, very nearly.

Put e = the length of the load line of the vessel, in feet;
f = the greatest immersed section, in square feet;
g = area of resistance in square feet;
h = displacement of the vessel in tons;
i = horse power required to propel the vessel j miles an hour (statute miles);
k = coefficient of the vessel;
l = resistance of the vessel in pounds;
m = tabular index corresponding to k;
n = slip of either propeller or paddle-wheels;
o = the pitch of the propeller;
p = the number of revolutions a minute.

EXAMPLES.

Ques. 88. Required the area of resistance of a vessel when $e=260$ ft.=the length of the load line; the greatest immersed section 500 square feet$=f$; the displacement of the vessel 2500 tons$=h$?

$$m=\frac{35\,h}{fe}=\frac{35\times2500}{500\times260}=\cdot67.$$

Opposite ·67 in the following Table, k will be found= 1·76, the coefficient of the vessel.

Then $g=\sqrt{\frac{f^3}{f+k\,e^2}}=\sqrt{\frac{(500)^3}{500+1\cdot76(260)^2}}=32$ square feet, the area of resistance required.

m.	*k.*	*m.*	*k.*
·50	1·31	·71	1·52
·51	1·37	·72	1·43
·52	1·45	·73	1·36
·53	1·51	·74	1·29
·54	1·56	·75	1·21
·55	1·62	·76	1·13
·56	1·66	·77	1·05
·57	1·71	·78	·978
·58	1·76	·79	·903
·59	1·83	·80	·835
·60	1·89	·81	·763
·61	1·93	·82	·691
·62	1·98	·83	·625
·63	2·01	·84	·559
·64	1·97	·85	·496
·65	1·91	·86	·433
·66	1·84	·87	·379
·67	1·76	·88	·326
·68	1·72	·89	·277
·69	1·64	·90	·229
·70	1·59	·95	·025

Ques. 89. Required the resistance of the vessel in pounds$=l$, when she is running at the rate of 10 miles an hour $=j$; the area of resistance $=40$ feet $=g$, found as in the last example.

The succeeding formula is often employed:

$l=4\cdot22\times gj^2=4\cdot22\times40\times100=16880$ lbs., the resistance of the vessel in pounds.

The resistance in pounds may be found by the following formula nearer the truth:

$l=4\cdot3\,gj^{1\cdot75}$; which, when put in dual logarithmic form, becomes

$$\downarrow l=\downarrow,(4\cdot3)+\downarrow,g+1\cdot75\downarrow,j$$

$$\begin{array}{lrl}
 & \downarrow,j=\downarrow,10=230258509, & \\
 & 3 & \\
 & 4)\overline{690775527} & \\
 & \overline{172693882,} & \\
(4\cdot3\times40=172.) & 54232429, & =\downarrow,(1\cdot72) \\
 & \overline{226926311,} & \\
 & 230258509, & \\
 & \overline{{}^{\prime}3332198} & =\downarrow,(\cdot96722708)
\end{array}$$

$\therefore\ l=9672\cdot3$, the first rule gave the resistance $=16880$ lbs.

Ques. 90. Required, the resistance of a vessel in lbs. (l), when she is running at the rate of 11·2 miles an hour (j); $e=200$ feet; $f=400$ square feet; the displacement of the vessel $=2000$ tons (h)?

$$m=\frac{35\,h}{fe}=\frac{35\times2000}{400\times200}=\cdot875.$$

From the table for $m=\cdot875$ the value of k is found to be $=\cdot353$, the coefficient of the vessel.

$$g=\sqrt{\frac{f^3}{f+\mathrm{K}\,e^2}}=\sqrt{\frac{400^3}{400+\cdot353\,(200)^2}}=66\cdot6 \text{ square feet};$$

Then, according to the second rule, the resistance in lbs. $=$

$$l=4\cdot3\,g\,j^{1.75}.$$

$4\cdot3\,g=4\cdot3\times66\cdot6=286\cdot38$, dual log. $=565731962$,

$$\begin{array}{rr}
\text{Dual logarithm of } j=\downarrow,(11\cdot2)= & 241591377, \\
241591377,\times(1\cdot75)= & 422784910, \\
\downarrow,(286\cdot38)= & \underline{565731962,} \\
 & 988516872, \\
\text{Dual log. of } 10^4 & \underline{921034037,} \\
\text{Dual log. of } 1\cdot96369603= & 67482845,
\end{array}$$

$$\therefore\quad l=19637 \text{ lbs. nearly.}$$

The dual logs. may be instantly calculated or found by merely inspecting the author's tables of dual logarithms, recently published.

Ques. 91. Required the horse power i, when the speed $=11$ knots an hour$=j$; and the area of resistance$=33$ square feet$=g$.

$$i=\frac{j^3\times g}{88}=\frac{11^3\times 33}{88}=499\tfrac{1}{8} \text{ horse power.}$$

The horse power may be found with greater accuracy by the succeeding logarithmic expression:

$$i=\frac{gj^{2.75}}{88}=\text{the horse power; or}$$

$$\downarrow,(i)=2\cdot75\downarrow,(j)+\downarrow,(g)-\downarrow,(88).$$

$$\frac{33}{88}=\frac{3}{8}=\frac{g}{88}=\cdot375.$$

$$\begin{array}{rrl}
\downarrow,(j)= & 239789527, & \text{Mult.} \\
 & \underline{2\cdot75} & \text{by} \\
 & 659421199, & \\
\downarrow,(\cdot375)= & \underline{'98082925} & \\
 & 561338274, & \\
 & \underline{460517018,}= & \downarrow,(10^2) \\
 & 100821256, &
\end{array}$$

100821256 = the dual logarithm of 2·7406977.

∴ The horse power = 274·07. (i).

The horse power, according to this latter method= 274·07; there is a discrepancy in these results also; the cube of the speed is employed in one case, and 2¾ power in the other.

$$\frac{j^{2\frac{3}{4}}\times g}{88},$$

put into a logarithmic form, gives the latter rule. Experiment has not yet settled this point in steam navigation, simple as it is; some employ the square of the velocity, others the cube, and a third party the 2¾ power. Government experiments, both in this country and elsewhere, are generally Government jobs. Engineers are selected to superintend them upon the principle that determined who should be the village schoolmaster; one of those worthies was chosen to fill the important office of village teacher merely because he had a large family and a wooden leg.

PROBLEM I.

To find the slip of screw propellers or paddle-wheels, when the acting area and area of resistance are given.

If the area of resistance=49 ft.=g; and the acting area =225 ft.=d; required the slip=n.

$$n=\frac{\sqrt{g^3}}{\sqrt{g^3}+\sqrt{d^3}}=\frac{343}{343+3375}=\frac{343}{3718}.$$

PROBLEM II.

To calculate the power and find the properties of paddle-wheels and screw-propellers.

The pitch of a propeller=33 ft., and makes 48 revolutions a minute, the slip =·35. What is the speed of the vessel?

$$\frac{33\times 48}{88}(1-\cdot 35)=11\cdot 7 \text{ knots.}$$

Put o=the pitch; p=the revolutions a minute; and n= the slip; then generally,

$$j=\frac{o\times p}{88}(1-n);$$

$$\therefore\quad p=\frac{88j}{o\times(1-n)};$$

$$\therefore\quad n=1-\frac{88j}{op}.$$

Let the speed be 11·7 knots the hour$=j$; the slip$=\cdot35=n$; the pitch 33 ft.$=o$; required the number of revolutions$=p$.

$p=\dfrac{88j}{o\times(1-n)}=\dfrac{88\times 11\cdot7}{33\times(1-\cdot35)}=48$, the number of revolutions a minute.

Required the slip, when the speed is 11·7 knots per hour, the revolutions 48 per minute, and the pitch=33 ft.

$$n=1-\frac{88j}{o\times p}=1-\frac{88\times 11\cdot7}{33\times 48}=\cdot35.$$

PROBLEM III.

Any four of the five following quantities being given, to find the fifth, namely, the radius of a paddle-wheel from the centre of motion to the centre of pressure of the floats $=c$; the slip$=n$; the immersed angle of the paddle-wheel at the centre of pressure$=b$; the statute miles per hour$=j$; and the revolutions a minute$=p$.

From the general expression

$$j=\frac{cp\cos.\frac{1}{3}b}{14}(1-n)$$

any one of the quantities is easily deduced, for

$$p=\frac{14j}{(1-n)\,c\cos.\frac{b}{3}};\quad n=1-\frac{14j}{pc\cos.\frac{1}{3}b}.$$

Suppose the immersed angle of the paddle-wheel at the centre of pressure$=54°\ 33'=b$; the slip$=\cdot38=n$; the revolutions per minute $=16=p$; and the radius of the paddle-wheel from the centre of motion to the centre of pressure of the floats $=21$ ft.; required the speed $=j$.

One-third of 54° 33′ = 18° 11′, the natural cosine of which is $=\cdot 9500629$; for practical purposes ·95 will be sufficiently near the truth.

$$j=\frac{c\,p\cos\frac{1}{3}b}{14}(1-n)=\frac{21\times 16\times\cdot 95}{14}(1-\cdot 38)=14\cdot 136 \text{ miles an hour.}$$

Again, let the immersed angle = 54° 33′; the revolutions per minute = 16; the speed 14·136 miles an hour; the radius of the paddle-wheel from the centre of motion to the centre of pressure = 21 ft.; required the slip.

$$n=1-\frac{14\times 14\cdot 136}{16\times 21\times\cdot 95}=1-\cdot 62=\cdot 38, \text{ the required slip.}$$

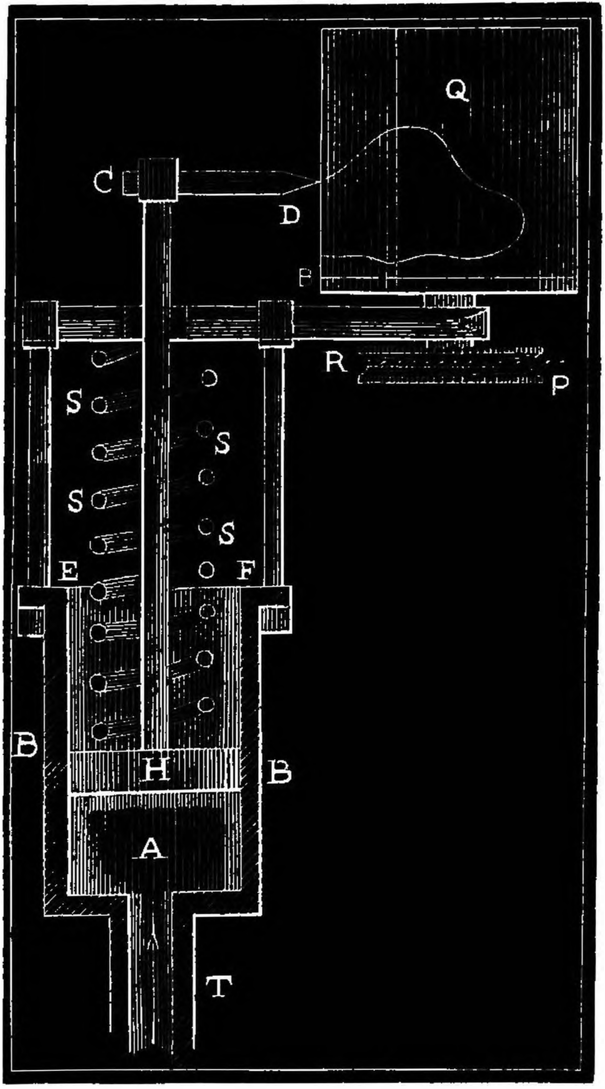
THE INDICATOR.

The *Indicator* is a simple and beautiful little instrument, by which the power of steam can be ascertained with great accuracy, when a properly constructed instrument is used; but we should know who employs it, and for what purpose it is employed. It can be made to indicate what is false as well as what is true. With very little cookery it may be made to show that a good engine is a bad one, and the contrary. There is a class of engineers, which shall be nameless, who make great use of this little instrument in cooking up their reports. Richards's steam-engine indicator, although not faultless, is the best in use.

The nature of this instrument is easily understood by referring to Fig. 145.

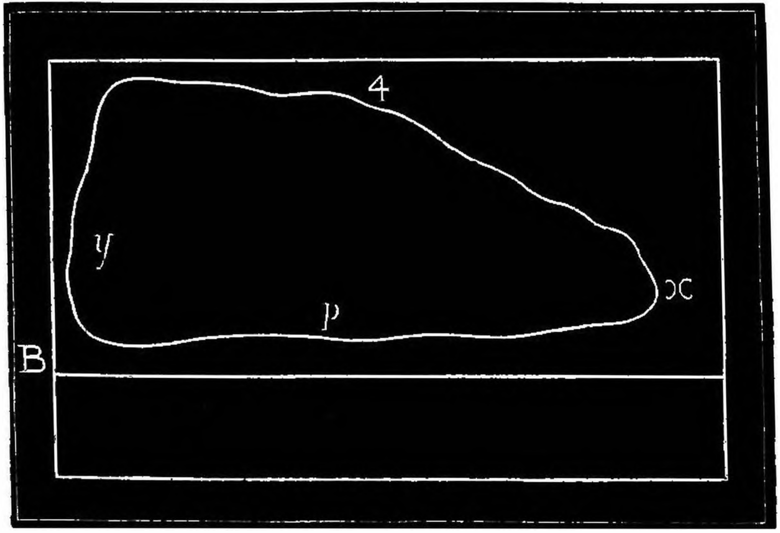
A is a small cylinder, B B open at top, E F fitted with a piston, and communicating with the lower part of the main steam cylinder by a pipe T, and stop-cock.

Fig. 145.



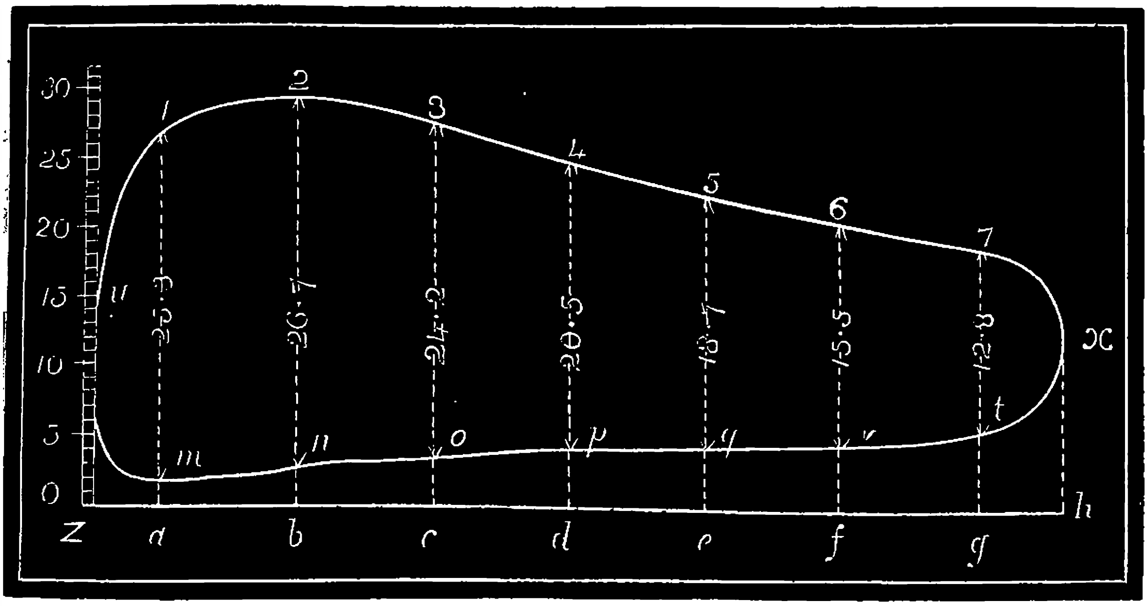
The piston, H, is pressed down into the cylinder by a nicely adjusted spiral spring, S S S, and a pencil, C D, is fixed to the piston-rod, H C; Q is a roller round which a piece of paper is wound; on the axis of this roller is fitted a pulley, P, connected by a string, P R, with some of the moving parts of the engine. The roller, Q, is also fitted with a spring, like the mainspring of a watch, in such a manner that, after being pulled round in one direction by the motion of the engine communicated through the string, R P, it is made to recoil by the spring. If we suppose the stop-cock in the pipe, T, closed, the piston, being pressed on by the spring S S S and the atmosphere, will remain at the bottom of the cylinder, B B ; and the pencil, being stationary at its lowest point, B, while the roller is made to rotate backwards and forwards, will describe a line on the paper which would appear straight on its being unfolded from the roller, D Q. But if, while the roller continues its motion, the stop-cock be opened, then the piston will be subjected to the pressure of steam in the main steam cylinder, and will be forced upwards in opposition to the pressure of the spring S S S and of the atmosphere ; and the pencil will trace a line on the paper varying in height as the piston H of the indicator rises and falls. But, further, if the spiral spring be so adjusted that we know exactly how many pounds will compress it an inch, and if we know the area of the piston H, we can measure the amount of pressure on it by the height to which the pencil is raised above the neutral line, B, where it remains when subjected to no upward pressure of the steam in the direction of the arrow ; and thus the position of the pencil on the paper, or the mark left by it at any point, furnishes the measure of the pressure on the main piston of the engine at the corresponding point of its stroke. On unfolding the paper from the roller Q, we should find a figure described on it by the pencil resembling *y*, 4, *x*, *p*, Fig. 146, which, when properly analysed, gives the means of

Fig. 146.



reckoning the varying pressures on the piston, and often points out defects in some of the adjustments, and suggests modes of remedying them. If we suppose that the area of the indicator-piston is one square inch, and that the spring is adjusted so that it requires a force of 10 lbs. to compress it 1 inch in length, or 1 lb. to compress it $\frac{1}{10}$th of an inch, we can form a scale of tenths of inches, and apply it to the indicator diagram at a number of points, *a*, *b*, *c*, &c. (Fig. 147), equally distant, and measure off the lengths of ordinates, *a* 1, *b* 2, &c., drawn through these points, and thus estimate the pressures acting on the piston of the indicator-cylinder at equidistant points of the stroke through which the paper is made to travel. These pressures correspond exactly with those to which the main piston of the engine has been subjected during its stroke, because the small cylinder, E B, of the indicator communicates freely with the cylinder of the engine. If we suppose the indicator to be fixed at the top of the steam cylinder, the upper part of the curve, 1, 2, 3, 4, 5, &c., is that traced during the descent of the piston when the steam is pressing on it. The lower part of the curve, *x*, *t*, *v*, *y*, is that traced during the ascent of the piston when the steam is escaping from the cylinder. Were the indicator fixed to the bottom of

Fig. 147.



the cylinder, we should get corresponding curves for the steam pressures there. Generally, when the slide gear is properly adjusted, these figures are very nearly alike, and, if so, the upper part of the curve, *y*, 1, 2, 3, 4, 5, 6, 7, *x*, may be taken as that traced by the active pressure either above or below the piston, while the lower part of the curve, *x*, *t*, *v*, *q*, *p*, *o*, *n*, *m*, *y*, may be taken as representing the corresponding resistance of steam during its egress from the cylinder. Now, as the total height, *c* 3, of any ordinate measures the total pressure on one side of the piston when it is at the point of its stroke corresponding to *c*, and as the part *c o* of the same ordinate represents the resisting pressure on the piston at the same point of its stroke, the difference *o* 3, or the part of the ordinate intercepted between the upper and lower boundaries of the curve, measures the effective pressure on the piston clear of all resistances. The same applies to all the ordinates, and, as we may suppose the whole curved space made up of numerous equal, narrow, vertical strips, each measured in height by an ordinate, we reckon the area of the figure contained within the curve as an expression of the power developed by the piston during its stroke; or, having taken a considerable number of pressure ordinates, and found the average, we consider this the mean effective pressure on the piston.

For example, the average of those marked on Fig. 147, found by adding them into one sum and dividing by their number, is

$$\frac{144{\cdot}9}{7} = 20{\cdot}7 \text{ lbs.}$$

the mean pressure on every square inch of the piston. In taking the average it is better to take the first and last ordinates, *m* 1, and *t* 7, at a distance from the respective ends of the stroke, $x\,y$ = half of *a b*, *b c*, *c d*, &c. If we suppose the engine from which Fig. 147 was taken to have a cylinder 20 in. diameter, 2 ft. 6 in. stroke, making 40 revolutions a minute, the power of the engine is readily calculated.

Area of piston $=20^2\times{}\cdot7854=314\cdot16$ square inches;

$314\cdot16\times20\cdot7=6503\cdot112$ mean pressure on the piston.

$$\begin{array}{rl} 6503\cdot112 & \\ 5 & \text{ft. double stroke} \\ \hline 32515\cdot560 & \\ 40 & \text{revolutions a minute} \\ \hline 1500622\cdot400 & \end{array}$$

$$\frac{1500622\cdot24}{33000}=45\cdot5 \text{ horse power, nearly.}$$

CALCULATION RESPECTING THE SCREW PROPELLER REPRESENTED IN FIG. 148.

This propeller has a particular expanding pitch.

Put $q=\frac{1}{4}$ the pitch at the periphery $=$ DE.

$r=\frac{1}{4}$ the pitch at the hub $=$ RS.

$s=$ HR, the assumed slip in a fraction of q.

$\therefore \quad q=r+s$.

Put $t=$ extreme radius $=$ OI.

$u=$ diameter of the screw $=2t=$ FI.

$v=$ pitch of the propeller at the periphery $=4q$.

$w=$ angle of the blades at the periphery $=$ KQT.

$w_1=$ angle of the blades at the hub.

JM $=x=\frac{1}{4}$ pitch of the propeller at the centre of effort of the blades.

BJ $=y=\cdot725t$, the radius at the centre of effort, G $=$ OG $=$ QM.

$\therefore$ The actual pitch at the centre of effort $=4x$.

$z=$ horse power required to drive the propeller and a revolutions a minute.

$\beta=$ number of blades.

$\gamma=$ length of blade parallel with the centre line $=$ UV.

$\delta=$ the breadth of the propeller blades over the edge between the corners XZ.

$\epsilon=$ the circular arc in the angle $\Theta=$ WOY.

$\Theta=$ the projected angle of the blades.

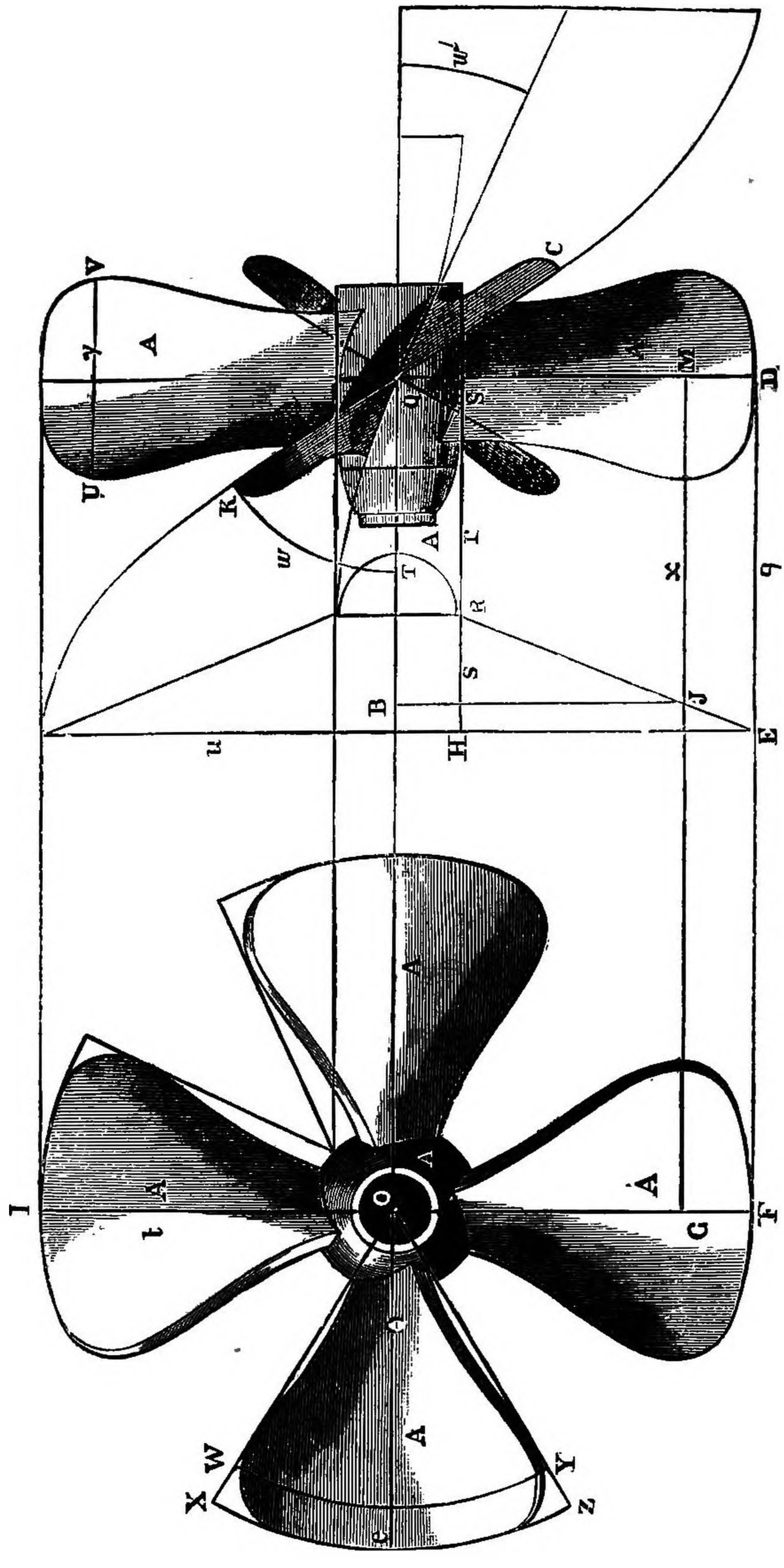
A $=$ the true inclined surface of the blades.

$A_1=$ the projected area of the blades.

$A_2=$ the acting area of the propeller.

Let $\pi=$ the circumference of a circle diameter $=1$.

Fig. 148.


V
γ
A
U
K
w
C
w'
Q
S
M
D
T
A
R
S
B
H
E
J
x
b
u
I
t
A
O
G
F
W
X
Y
Z
c

$\text{cot. } w=\frac{v}{\pi u}=\frac{4q}{\pi\, 2t}=\frac{2q}{\pi t}$, the cotangent of the angle of the blades at the periphery.

$v=\pi u \text{ cot. } w$, the pitch of the propeller at the periphery.

$$v=\frac{360}{\theta}\times\gamma;$$

$$\theta=\frac{360}{v}\times\gamma$$

$$v=\frac{\pi u \gamma}{\epsilon};\ v=\frac{\pi u \gamma}{\sqrt{\delta^2-\gamma^2}};\ \theta^2=\frac{129600\gamma^2}{v^2};$$

δ is the hypothenuse, and γ is the base of a right-angled plane triangle.

$\delta=$ KC measured over WY, it is a projection the same as UV $=\gamma$.

$$\delta=\sqrt{\frac{\theta^2\pi^2u^2}{129600}-\gamma^2}$$

$$A_1=\frac{\cdot 7854\, u^2\,\theta\,\beta}{360}$$

$$A=\frac{4t\,\beta(\delta+\gamma)}{9}.$$

$$A_2=\frac{5u^3}{2\sqrt{u^2+\pi^2u^2}}$$

$$z=\frac{u^3a^3}{480000}(\gamma\, s\text{cos. } w+\frac{1}{9})$$

PROBLEM IV.

The pitch of a propeller $=33$ ft., and makes ·48 revolutions a minute, the slip $=\cdot 35$. What is the speed of the vessel?

$$\frac{33\times 48}{88}(1-\cdot 35)=11\cdot 7 \text{ knots.}$$

By the introduction of other screws the slip might be reduced from 35 to 15 per cent., the speed, when the slip is 15 per cent., will be

$$\frac{33\times 48}{88}(1\cdot-\cdot 15)=15\cdot 3 \text{ knots.}$$

Put $o=$the pitch;
$p=$the revolutions a minute;
$n=$the slip; and
$j=$the knots the hour;

then generally,

$$j=\frac{o\times p}{88}(1-n);$$

$$p=\frac{88j}{o\times(1-n)};$$

$$n=1-\frac{88j}{o\times p};$$

$$o=\frac{88j}{p\,(1-n)}.$$

A ship fitted up with a screw propeller made 13 knots an hour; the revolutions were 60 a minute, and the pitch 30 ft. What was the slip?

$$n=1-\frac{88j}{o\times p}=1-\frac{88\times 18}{30\times 60}=\cdot 12, \text{ or 12 per cent.}$$

PROBLEM V.

Given the diameter of a propeller$=12$ ft.$=w$; and the angle at the periphery$=51^\circ\,..\,33'=w$; to find the pitch $=v$.

$$\text{cot. } 51^\circ\,..\,33'=\cdot 7940121$$
$$v=\pi\, u \text{ cot. } w=3\cdot 1416\times 12\times\cdot 794=29\cdot 93 \text{ ft.}$$

PROBLEM VI.

Given the diameter$=21$ ft.$=u$; the length of the blade parallel with the centre line$=6$ ft.$=y$; the slip 40 per cent., or $S=40$; the angle of the blades at the periphery$=63^\circ\,..\,42'=w$; to find the horse power necessary to drive this screw 68 revolutions a minute$=a$.

$$\text{Cosine } 63^\circ..\ 42' = \cdot 4430712$$

$$z = \frac{21^3 \times 68^3}{480000}\left(6 \times \cdot 4 \times \cdot 443 + \frac{1}{9}\right) = 7124.$$

$$6 \cdot \times \cdot 4 \times \cdot 443 + \frac{1}{9} = 1 \cdot 17431111$$

$$21 \times 68 = 1428.$$

$$\begin{array}{rll}
\downarrow, (1\cdot 428) = & 35627490, & \\
 & \underline{\qquad\qquad 3} & \\
 & 106883470, & \\
\downarrow, (1\cdot 17431111) = & \underline{16068179,} & \\
 & 122951649, & \\
\downarrow, (\cdot 48) & \underline{{}'73396917} & \text{subtract} \\
 & 196348566, & \\
 & \underline{{}'230258509} & \downarrow, (\tfrac{1}{10}) \\
\downarrow, (\cdot 71241162) = & {}'33909943 &
\end{array}$$

$$\therefore \quad z = 7124 \cdot 1162, \text{ as before.}$$

1·428 mult. by 1000. = 1428.

·480000 mult. by 1000000 = 480000.

$$(1000)^3 = 10^9 \text{ and } \frac{10^9}{10^6} \times 10 = 10^4.$$

Hence the decimal point had to be moved four places to the right, which brought ·71241162 to 7124·1162.

As a second example, suppose $u = 12$ feet; $\gamma = 5$ feet; $w = 57^\circ\ 30'$; $s = 38$ per cent., or $s = \cdot 38$; what power is required to drive this propeller 40 revolutions a minute?

$$z = \frac{15^3 \times 40^3}{480000}\left(5 \times \cdot 38 \times \cdot 5373 \times \tfrac{1}{9}\right) = 509 \cdot 4.$$

Calculation by Dual Logarithms.

$$15 \times 40 = 600.$$

$$5 \times \cdot 38 \times \cdot 5373 + \tfrac{1}{9} = 1 \cdot 13198111.$$

$$\begin{array}{llll} \downarrow,(\cdot6)= & '51082562 & & \\ & \underline{\qquad\qquad 3} & & \\ & '153247686 & '153247686 & \\ \downarrow,(1\cdot13198111)= & 12396933, & \bar{1}87603067 & \text{ar. co.} \\ \text{subtract } \downarrow,(\cdot48) & '73396917 & \bar{1}26603083 & \\ & \downarrow,(\cdot5093915) & '67453836 & \end{array}$$

∴ $z=509\cdot4$, found by adding three dual numbers together.

PROBLEM VII.

Given the diameter of a propeller $=21$ feet $=u$; the angle of the blades at the periphery $=63°\,..\,42'=w$; the length $\gamma=6$ feet; the slip $s=\cdot40$ or 40 per cent.; four engines drive this screw of 7124 horse power $=z$; how many revolutions, α, will this propeller make in a minute?

$$\alpha=\frac{1}{u}\left\{\frac{480000z}{(\gamma\, s \cos.\, w+\frac{1}{9})}\right\}^{\frac{1}{3}}=68 \text{ revolutions.}$$

$$\gamma\, s \cos.\, w+\tfrac{1}{9}=6\times\cdot4\times\cdot443+\tfrac{1}{9}=1\cdot1743.$$

By Dual Logarithms.

$$\begin{array}{lcll} \downarrow,(\cdot480000) & = & '73396917 & \rbrace \text{ add.} \\ \downarrow,(.7124) & = & '33909943 & \\ \downarrow,(1\cdot17431) & = & \underline{16068179,} & \text{subtract.} \\ & & '123375039 & \\ & & \underline{'460517019} & \downarrow,(100.) \\ & & 3)\underline{'583892058} & \\ \downarrow,(2\cdot1) & & '194630686 & \\ & & \underline{74193735,} & \\ & & '268824421 & \\ & & \underline{230258509,} & \downarrow,(10.) \\ \downarrow,(\cdot68000231)= & & '38565912 & \\ & \therefore & \alpha=68. & \end{array}$$

PROBLEM VIII.

Given, the diameter of the propeller $=13$ feet $=u$; the angle of the blades at the periphery $=59°\ 46'=w$; the angle of the blades at the hub $= 11° \, 48' = w_1$; the diameter of the hub $= 1{\cdot}25$ feet $=$ twice QS; (see Figure 148) to find the pitch at the centre of effort.

Cotangent of 59° 46′ $= {\cdot}582793$
Cotangent of 11° 48′ $= 4{\cdot}786730$

The pitch at the periphery $=$

$$13 \times {\cdot}583 \times 3{\cdot}1416 = 23{\cdot}8 \text{ ft.} = v$$

The pitch at the hub $=$

$$1{\cdot}25 \times 4{\cdot}787 \times 3{\cdot}1416 = 18{\cdot}8 \text{ ft.} = v_2$$

Put v_1 for the pitch at the centre of effort, u_1 the diameter at the centre of effort, and u_2 the diameter at the hub.

$$(v - v_2) : (v_1 - v_2) :: (u - u_2) : ({\cdot}725u - u_2)$$

$$\therefore \quad v_1 = v_2 + \frac{(v - v_2)\,({\cdot}725u - u_2)}{u - u_2}$$

$$v_1 = 78{\cdot}8 + \frac{5 \times 8{\cdot}175}{11{\cdot}75} = 22{\cdot}3$$

Recapitulation.
$v = 23{\cdot}8$ ft. $\quad u = 13$ ft.
$v_1 = 22{\cdot}3$ ft. $\quad u_1 = {\cdot}725u = 9{\cdot}425$ ft.
$v_2 = 18{\cdot}8$ ft. $\quad u_2 = 1{\cdot}25$ ft.

PROBLEM IX.

Given, the diameter of a propeller $= 13{\cdot}5$ feet $= u$; and the angle $w = 65°$, to find the acting area of the propeller.

$$A_2 = \frac{5u^3}{2\sqrt{v^2 + \pi^2 u^2}};$$

but $v^2 = \pi^2 u^2 \cot.^2 w$, hence

$$A_2 = \frac{5u^3}{2\pi u \text{ cosec. } w}.$$

Calculation by Dual Logarithms.

Dual log. sine of 65° = ’9837630
„ „ cosine of 6·5° = ’86128586
76290956 difference.

☞ The student's particular attention to the facts which we shall enumerate: when we have the dual log., sine log., cosine, and the difference, we have at the same time the dual log., tangent log., cotangent log., secant, and log. cosecant

65°	Sine	'9837630	dual logarithms.
	Cosine	'86128586	
	Tangent	76290956,	
	Cotangent	'76290956	
	Secant	86128586,	
	Cosecant	9837630,	

These numbers are all whole numbers; if '9837630 be considered positive, 9837630, by changing the position of the comma, becomes negative; and on the contrary, if 9837630 be positive, '9837630 must be taken as negative. These remarks apply to dual logarithms of all sorts.

$$\begin{array}{rr} \text{Log. } 1{\cdot}35 = & 30010460, \\ \text{log. } \pi = & 114472989, \\ \text{log. cosecant } w = & \underline{9837630,} \\ & 154321079, \end{array} \qquad \begin{array}{rr} \text{log. } 2{\cdot}5 = & 91629072, \\ 3 \text{ log. } 1{\cdot}35 = & \underline{90031380,} \\ & 181660452, \\ & \underline{154321079,} \\ \text{log. of } 1{\cdot}31441767 = & 27339373, \end{array}$$

∴ Acting area of the propeller, Fig. 148, = 131·1441767 square feet = A_2.

The screw propeller, Fig. 148, is selected, like our other examples, for no particular excellence, but merely for the purpose of illustration. The examples here introduced show the great importance of dual logarithms, which are easily calculated when tables are not at hand. Conventional arrangements may indicate processes precisely, and yet render little or no assistance to an operator trying to obtain results. This grave objection applies to many of our modern mathematical researches and formulæ without the intervention of dual arithmetic.

CHAPTER X.

USEFUL ARITHMETICAL PROCESSES, OPERATIONS THAT MAY BE FORGOTTEN, AND THE APPLICATION OF PRINCIPLES OFTEN MISUNDERSTOOD BY THOSE WHO HAVE NEGLECTED THE STUDY OF ARITHMETIC.

ARITHMETIC is the science of numbers and their notation, and of the different operations to which numbers are subjected. Arithmetic explains and illustrates practically the methods of computing by numbers. At present there are two systems of arithmetic, namely, common and dual—the latter invented and developed by the author of the present work. When numbers are very great, for the purpose of facilitating numeration, they are generally divided into periods of six figures each, the first on the right hand being called the units' period, the next the period of millions, the third the period of billions, the fourth the period of trillions, and so on, as in the following example:

Billions.	Millions.	Units.
693147	180230	258509,

which is read, six hundred and ninety-three thousand one hundred and forty-seven billions, one hundred and eighty thousand two hundred and thirty millions, two hundred and fifty-eight thousand five hundred and nine.

In France, and other Continental places, numbers are divided into periods of three figures each; the first on the right hand is called units' period, the next the period of thousands, the third the period of millions, and so on. According to this system of notation and numeration, the preceding number will be expressed thus:

Quadrillions.	Trillions.	Billions.	Millions.	Thousands.	Units.
693	147	180	230	258	509.

Both methods agree in all numbers not exceeding nine places of figures, but afterwards the French plan is more simple than the English. The student is supposed to understand addition, subtraction, multiplication, long division, and short division of whole numbers.

VULGAR FRACTIONS.

When we divide a quantity into several equal parts, one or more of them is called a fraction of the quantity so divided; thus, if a foot be divided into 12 inches, and 5 of these inches taken, we will have what is termed *five-sevenths* of a foot, written thus, $\frac{5}{7}$ of a foot. The number of parts into which the quantity is divided is called the *denominator*, and is put below the line which is placed between the numbers; and the number of parts taken is called the *numerator*, and is put above the line which is placed between the numbers composing the fraction.

The numerator we shall sometimes call the number *above*;
The denominator „ „ „ „ *below*.

To multiply a fraction by a number, multiply the number above the line, but retain the number below unchanged. *Example.* $\frac{3}{16}$ multiplied by $9 = \frac{27}{16}$. To divide a fraction by a number, multiply the number below, and retain the upper number unchanged. *Example.* $\frac{5}{6}$ divided by $3 = \frac{5}{18}$. If a fraction be multiplied above and below by any number, its value is not altered. *Example.* The fraction $\frac{2}{7}$ multiplied above and below by any number, as 5, gives $\frac{10}{35}$; then $\frac{2}{7}=\frac{10}{35}$; also,

$$\frac{2}{7}=\frac{4}{14}=\frac{6}{21}=\frac{8}{28}=\&c.$$

On the contrary, if a fraction be divided *above* and *below* by the same number, its value is not altered. *Example.* $\frac{20}{24}$ divided *above* and *below* by 2 gives $\frac{10}{12}$; then $\frac{20}{24}=\frac{10}{12}=\frac{5}{6}$.

Multiplication of Vulgar Fractions.

Rule. Multiply the upper numbers together for the numerator, and the lower numbers together for the denominator of the product.

Example. $\frac{3}{4}$ multiplied by $\frac{5}{7}=\frac{15}{28}$, the product.

Division of Vulgar Fractions.

Rule. Invert the divisor, then multiply the numbers as in multiplication.

Example. $\frac{2}{3}$ divided by $\frac{7}{11}$; $\frac{7}{11}$ inverted becomes $\frac{11}{7}$;

$$\text{Then } \frac{2}{3} \text{ and } \frac{11}{7} \text{ gives } \frac{22}{21}=1\tfrac{1}{21}.$$

Addition of Vulgar Fractions.

Rule. Multiply the numbers composing the two fractions crosswise, add the two products together, under which place the product of the numbers below the lines of the given fractions.

Example. Add $\frac{3}{5}$ and $\frac{4}{7}$ together.

$$\frac{3}{5}\times\frac{4}{7}$$

$$\begin{array}{rl} & 20=5\times 4 \\ & 21=7\times 3 \\ \text{Sum} & \overline{41} \\ & \overline{35}=7\times 5 \end{array}$$

$$\therefore\quad \frac{3}{5} \text{ added to } \frac{4}{7}=\frac{41}{35}=1\tfrac{6}{35}.$$

Subtraction of Vulgar Fractions.

Rule. Multiply the numbers composing the two fractions crosswise, take the difference of these products, under which place the product of the numbers below the lines of the given fractions.

Example. From $\frac{3}{5}$ take $\frac{4}{7}$.

$$\frac{3}{5}\times\frac{4}{7}$$

$$\begin{array}{l} 21=7\times 3 \\ 20=5\times 4 \\ \overline{1} \\ \overline{} \\ 35=5\times 7 \qquad \therefore\ \text{Difference}=\frac{1}{35}. \end{array}$$

In the fraction $\frac{2}{3}$ we suppose 1 to be divided into 3 equal parts, and 2 of these parts to be taken; but we might suppose the number 2 to be divided into 3 parts, and one of these parts to be taken, and the result would be exactly the same. Suppose the unit to be a foot, then $\frac{1}{3}$ of a foot $=$ 4 inches; therefore $\frac{2}{3}$ of a foot $=8$ inches. But the $\frac{1}{3}$ of

2 feet must be exactly the same = 8 inches. The same may be said of other fractions.

$\frac{4}{5}$ of an inch $=\frac{4}{60}$ of a foot; for $\frac{4}{5}$ divided by $12=\frac{4}{60}=\frac{1}{15}$.

$\frac{3}{8}$ of a cubic foot$=\frac{3\times1728}{8}=648$ cubic inches; for $\frac{3}{8}$ multiplied by $1728=\frac{5184}{8}=648$.

To divide a whole number and a fraction by a whole number.

Divide $23\frac{4}{5}$ by 7.

$$\begin{array}{r} 7\underline{)23\frac{4}{5}} \\ 3\frac{14}{35}=3\frac{2}{7} \end{array}$$

7 into 23 goes 3 times and 2 over, 5 times 2 = 10 and 4 gives 14; and 5 times 7 = 35, gives $3\frac{14}{35}$.

$\frac{14}{35}=\frac{2}{7}$; dividing above and below by 7.

THE RULE OF THREE.

Although it may be considered a digression from our general plan to treat of the rule of three any further than to show how to perform its calculations, yet it is necessary to treat of it at some length, from the very defective manner in which it has hitherto been handled by writers on common arithmetic. Most of them divide it into three separate divisions, namely, "the Single Rule of Three Direct," "the Rule of Three Inverse," and "the Double Rule of Three;" but here it is considered as one simple rule.

It is named "the Rule of Three" because there are generally three numbers given to find a fourth. From its extraordinary utility it is called "the Golden Rule;" again, "the Rule of Proportion," which is the most proper appellation. It is necessary to observe the following *Remarks*, in order to elucidate this method:

First. There is but one number given in a question in proportion that is of the same name or denomination as the answer or number sought.

Second. This number may be a compound one, but can always be reduced; for example:—Suppose 120*l.* to be paid for 8 packs of canvas, each pack containing 4 pieces, each piece 20 yards; how many yards, at the same rate, can be purchased for 24*l.*? As the number of yards in the 8 packs can be readily found, the question may stand thus:

Ques. 1. Suppose 120*l.* be paid for 640 yards, how many yards will 24*l.* buy at that rate?

Third. When a question is given in proportion, it is sometimes necessary to reduce the compound terms, and then write it down again in its contracted state.

Example.

Ques. 2. If 12 men in 4 days, when the days are 12 hours long, can dig a ditch 20 yards long, 4 deep, and 5 wide, how many men will be required to dig another, 240 yards long, 8 wide, and 7 deep, in 18 days, when the days are 16 hours long? This question, when reduced, will stand simply thus:—If 12 men in 48 hours excavate 400 cubic yards of earth, how many will be necessary to excavate 13440 cubic yards in 288 hours?

Fourth. When a question is thus reduced to its lowest terms, and the number which is of the same name or denomination as the one sought be selected, the remaining numbers must, *two* by *two*, express quantities of the same kind, or they are not to be in the proportion.

Example.

Ques. 3. If 12 men, 6 feet high, and 13 stone weight, excavate 400 cubic yards of earth in 48 hours, how many men will it require to excavate 13440 cubic yards in 288 hours?

	Hours.	Cubic yds.	Men.
6 feet	48	400	12
13 stone	288	13440	

Thus, as it is a number of men that is sought, 12 is selected, and is to be in the proportion; so are 400 and 13440, as both express cubic yards; and also 48 and 288, as both express hours; but 6 and 13 must be left out, as they stand for magnitudes of different kinds.

Fifth. When the numbers given are reduced, if any two of them which express similar magnitudes are equal, they may be left out of the proportion.

Example.

Ques. 4. If 32 *cwt.* be carried 40 miles for 16*l.* 12s., how far will 24 *cwt.* be carried for 332 shillings?

	Shil.	*Cwt.*	*Miles.*
£16 12s. =	332	32	40
	332	24	

From this it is evident that 32, 24, and 40 are to be in the proportion; but 16*l.* 12s. and 332 shillings may be omitted.

Sixth. When the necessary numbers for the proportion are selected, and the superfluous ones expunged, set down in a convenient place the number that expresses the magnitude which is the same as the one sought, and the remaining necessary numbers arranged *two* by *two*; then state the question:—Suppose two of the numbers which express magnitudes of the same kind, in existence, and the others equal, or not in existence, and the one which is like that required; then put the question whether the one sought is to be greater or less than the given one; if greater, the greater of the two is to be the centre term, and the other the first. In the same manner, the other magnitudes which are of the same kind are disposed of. Then multiply the first terms together, and divide the product into the continued product of the second and third terms for the answer.

Examples.

Ques. 5. Suppose 120*l.* to be paid for 640 yards of cloth, how many yards can be bought for 24*l.*?

A, or first term ;		B, or second ;		C, or third term.
£		£		*Yds.*
120	:	24	: :	640

In this question there are but three numbers given to find a fourth; their positions are thus found:—As the number sought is yards, the yards given are to be the third term, C; then the question is put—If 120*l.* buy 640 yards, will 24*l.* buy more or less? It will evidently buy less; therefore the less of the remaining numbers must be the second term, B, and the other the first term, A. When the question is thus stated, multiply B and C together, and divide the product by A for the number of yards sought.

WORK.

```
   £        £       Yds.
  120   :   24  ::  640
                     24
                   ----
                   2560
                  1280
                  -----
          12|0)1536|0
               ----
                128 yards, answer.
```

Ques. 6. If 32 *cwt.* be carried 40 miles for 16*l.* 12s., how far will 24 *cwt.* be carried for the same money?

The 16*l.* 12s. is superfluous, for reasons before given; the remaining numbers are necessary, and must be in the proportion, which will stand thus:

Cwt.		*Cwt.*		*Miles.*
24	:	32	: :	40

For miles are required, therefore 40 miles is the third; and if we put the question—If 32 *cwt.* be carried 40 miles for a certain sum, will 24 *cwt.* be carried a greater or a less distance for the same money? It is evident that it is a greater number than 40 that is required; consequently, the greater of the two remaining numbers is put for the second term, and the other for a first.

The work will stand thus:

Cwt.		*Cwt.*		*Miles.*
24	:	32	: :	40

40

24)180 (53⅓ miles, answer.

120

80

72

$\frac{8}{24}=\frac{1}{3}$

Ques. 7. If 12 men in 48 hours excavate 400 cubic yards of earth, how many will be necessary to excavate 13440 cubic yards of earth in 288 hours?

Hours.		Hours.		Men.	
288	:	48	: :	12	The magnitude which is of the same denomination as the one sought.

Cub. Yds.		Cub. Yds.
400	:	13440

The above stating is ascertained as follows:—As it is men that are sought, the 12 which expresses men is set down first; then suppose the cubic yards equal, to find the position of 48 and 288, say—If a certain number of cubic yards be excavated by 12 men in 48 hours, will it take a greater or less number of men to excavate the same number of yards in 288 hours? It is evident that a less number of men will be required; therefore, the less of the two numbers (48) is to be in the centre, and the other first. Then to determine the positions of 400 and 13440, say—If 400 yards be excavated by 12 men, will it take a greater or less number of men to excavate 13440 yards? It is evident that it will take a greater number; therefore, the greater of the two (13440) is to be in the centre, and the other (400) first.

WORK.

Hours.		Hours.		Men.
288	:	48	: :	12
Cub. Yds.		Yds.		
400	:	13440		
115200		48		

$$
\begin{array}{r}
13440\\
48\\
\hline
107520\\
53760\ \ \\
\hline
645120\\
12\\
\hline
11520|0)\ 774144|0\ (67\tfrac{1}{5}\ \textit{men, answer.}\\
69120\ \ \ \ \\
\hline
82944\ \ \\
80640\ \ \\
\hline
2304\\
\hline\hline
11520 = \tfrac{1}{6}
\end{array}
$$

Decimal Fractions.

We have before stated that a fraction expresses part of any whole thing by two numbers, one placed above a line and the other below it; and we have also stated, that the number below the line, in this conventional arrangement, is called the *denominator*, and shows how many parts the quantity is to be divided into; the number above the line is called the *numerator*, and specifies the number of such parts to be taken. For example, let us take the following fractions,

$$\frac{1}{2};\ \frac{3}{4};\ \frac{2}{3},\ \text{and}\ \frac{4}{7}\ \begin{array}{l}\text{numerators,}\\ \text{denominators;}\end{array}$$

which are called one-half, three-fourths, two-thirds, and four-sevenths respectively. When the *numerator* and *denominator* are equal, as $\frac{7}{7}$, the fraction is = 1; and when the numerator is greater than the denominator, the fraction is called *improper*, and is greater than 1; $\frac{9}{7}$, which is equal $\frac{7}{7}$ and $\frac{2}{7}$ together, or equal $1\frac{2}{7}$. Since, as before remarked, if the numerator and denominator of any fraction be multiplied or divided by any number, it neither increases nor diminishes the value of the fraction; then

$$\frac{1}{2}=\frac{2}{4}=\frac{4}{8}=\frac{40}{80}=\frac{400}{800}=\&c.,$$

$$\text{and } \frac{40}{80}=\frac{20}{44}=\frac{2}{4}=\frac{1}{2};$$

for it is evident that if a quantity be divided into 800 equal parts, and 400 of them taken, one-half the quantity is taken; the same reasoning holds good with other fractions.

$$\frac{3}{4}=\frac{30}{40}=\frac{300}{400}=\frac{3000}{4000}=\&c.,$$

$$\frac{4}{7}=\frac{40}{70}=\frac{400}{700}=\frac{4000}{7000}=\&c.;$$

so that it appears a fractional part of anything may be expressed in an infinite number of ways.

From the above it appears that one fraction may be reduced to another equal to it, and having, either the numerator or denominator, what number we please; for example, let us reduce $\frac{2}{3}$ to a fraction equal to it whose denominator shall be 1000.

$$\frac{2}{3}=\frac{2000}{3000} \text{ and } \frac{2000}{3000}=\frac{666\frac{2}{3}}{1000} \quad \therefore \frac{2}{3}=\frac{666\frac{2}{3}}{1000}$$

The reduction of vulgar fractions to decimals, is nothing more than the reduction of one fraction to another whose denominator will be 10, 100, 1000, 10000, &c., then cancelling the denominator, and placing a full point to the left of the numerator, which must consist of as many places of figures as there are ciphers cancelled in the denominator; any deficiency of figures in the numerator must be made up with ciphers to the left.

EXAMPLES IN REDUCTION.

Ex. 1. Reduce $\frac{3}{4}$ to a decimal.

$$\frac{3}{4}=\frac{300}{400} \text{ and } \frac{300}{400}=\frac{75}{100} \quad \therefore \quad \frac{3}{4}=\frac{75}{100}, \text{ or } \cdot 75 \text{ of a decimal.}$$

CONTRACTED.

```
4)30000
   ·75   answer.
```

Ex. 2. Reduce 15s. 7d. to the decimal of a pound sterling.

15s. 7d.=187 pence.
1*l*. =240

Then the vulgar fraction is expressed by $\frac{187}{240}$, the decimal is ·7791667 to seven places of figures; this is far enough to continue decimals for most practical purposes, and when the last figure is 5, 6, 7, 8, or 9, the figure preceding may be counted one more.

WORK.

```
24|0) 187      (77916666, &c.
      168
      190
      168
       220
       216
         40
         24
         160
         144
          16, &c.
```

IMPORTANT RULE.

☞ When the last figure and all that follow are rejected, and when the last figure is 5, 6, 7, 8, or 9, the figure preceding may be counted one more; ∴ the decimal of $\frac{187}{240}$ to five places is ·77917.

Ex. 3. Reduce $\frac{2}{63}$ to a decimal fraction.

63) 200000000 (·03174603
189
110
63
470
441
290
252
380
378
200
189
11

In this division there are eight ciphers used, then there must be eight places of figures in the quotient, and therefore one cipher is to be placed to the left of the figures obtained, and then the full point; or, which is the same, for every cipher added to the numerator there should be a cipher or some figure placed in the quotient, and before the figure put in the quotient for the first cipher added, the full point.

EXAMPLES FOR PRACTICE.

Ex. 4. Reduce 2 ft. $7\frac{1}{2}$ in. to the decimal of a yard.
Ans. ·875.

Ex. 5. What decimal is equal to $\frac{5}{61}$?
Ans. ·081967 and 13 over.

Ex. 6. Reduce $17\frac{3}{7}$ to a mixed decimal.
Ans. 17·4285714, &c.

TO FIND THE VALUE OF ANY DECIMAL.

Rule.—Multiply the given decimal by any denomination less than it, pointing off as many decimals in the product as are in the given decimal; then the figure or figures to the left of the separating point will be the number of that denomination contained in the decimal. Where the decimal has to be reduced to two or more denominations, the process may be repeated, as in the following

EXAMPLES.

Ex. 1. Required the value of ·11111111 of a hogshead containing 63 gallons.

$$\begin{array}{r} \cdot 11111111 \\ 63 \\ \hline 33333333 \\ 66666666 \\ \hline 6\cdot 99999993 \end{array}$$

When ·999. . . . follows the decimal point, the whole number may be increased by one; therefore the answer is 7 gallons.

Ex. 2. What is the value of ·87625 of a pound sterling?

$$\left.\begin{array}{r} \cdot 87625 \\ 20 \\ \hline 17\cdot 52500 \\ 12 \\ \hline 6\cdot 30000 \\ 4 \\ \hline 1\cdot 20000 \end{array}\right\}$$

∴ the value of 8·7625*l.* is equal 0*l.* 17s. 6$\frac{1}{4}$d. and $\frac{2}{10}$ of a farthing over.

Ex. 3. Find the value of ·087648 of a yard.

$$\left.\begin{array}{lr} & \cdot 087648 \\ & 3 \\ \hline \text{feet} = & 0\cdot 262944 \\ & 12 \\ \hline \text{inches} = & 3\cdot 155328 \\ & 12 \\ \hline \text{parts} = & 1\cdot 863936 \end{array}\right\}$$

0·87648 of a yard=3 inches 2 parts nearly.

EXAMPLES FOR PRACTICE.

Ex. 4. Required the value of ·74325 of a pound sterling.
Ans. 0*l.* 1*s.* 5$\frac{3}{4}$*d.* 352.

Ex. 5. Reduce ·8383838 of a foot to its equivalent value of inches and parts.

Ans. 10 in. 0·7272 part.

Ex. 6. What is the value of ·2648125 of a mile?

Ans. 2 fur. 4 pol. 4 yds. 0 ft. 2 in. 1·56 parts.

Most civilised nations employ the metric system of weights and measures.

ADDITION OF MIXED NUMBERS AND DECIMAL FRACTIONS.

Rule.—Keep the decimal points under each other, and proceed as in common addition, observing to point off as many decimals in the sum as there are in the term which contains the greatest number of decimals;

OR,

As placing ciphers after decimals does not increase or diminish their value, affix ciphers until each of the given terms consists of the same number of decimals, then proceed as in common addition, pointing the sum as before.

EXAMPLES.

Ex. 1. Required the sum of 3·46; 21·347; 1·3; and 24·0049.

$$\begin{array}{r|r}
 & \text{or thus,} \\
3{\cdot}46 & 3{\cdot}4600 \\
21{\cdot}347 & 21{\cdot}3470 \\
1{\cdot}3 & 1{\cdot}3000 \\
24{\cdot}0049 & 24{\cdot}0049 \\
\hline
50{\cdot}1119 & 50{\cdot}1119
\end{array}$$

The reason of this rule is obvious, for

$$\begin{aligned}
3{\cdot}46 &= 3\tfrac{46}{100} = 3\tfrac{4600}{10000} \\
21{\cdot}347 &= 21\tfrac{347}{1000} = 21\tfrac{3470}{10000} \\
1{\cdot}3 &= 1\tfrac{3}{10} = 1\tfrac{3000}{10000} \\
24{\cdot}0049 &= 24\tfrac{49}{10000} = 24\tfrac{49}{10000} \\
\hline
&\text{Sum} = 49\tfrac{11119}{10000} =
\end{aligned}$$

$50\tfrac{1119}{10000} = 50{\cdot}1119$, the same as above.

EXAMPLES FOR PRACTICE.

Ex. 2. What is the sum of 0·075; 11·0712; 171·; 47·1433 and 145·03215.

Ans. 374·32165.

Ex. 3. Reduce the following fractions to decimals, and find their sum; namely, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{10}$, $\frac{2}{11}$, $\frac{8}{25}$, and $\frac{7}{12}$.

Ans. 1·7684848.

SUBTRACTION OF DECIMALS.

Rule.—Place the numbers under each other as in addition, affixing ciphers, if necessary, to make the decimal places equal; then proceed as in common subtraction, pointing off the decimals in the remainder as in addition.

EXAMPLES.

What is the difference between 36·47 and 19·3764; 38·4 and 3·84; and 341·365815 and 276·82.

	Ex. 1.	*Ex.* 2.	*Ex.* 3.
From	36·4700	38·40	341·365815
Take	19·3764	3·84	276·820000
Remainder	17·0936	34·56	64·545815

Ex. 4. Required the difference between 8·75+·825 and 5·681+2·17.

Ans. 1·724.

MULTIPLICATION OF DECIMALS.

Rule.—Multiply the factors as in common multiplication; then point off as many decimals in the product (from the right) as there are in both factors.

☞ When it happens that the figures in the product are less than the decimals in both factors, ciphers must be prefixed to the left of the product to make up the deficiency. ☜

EXAMPLES.

Ex. 1.

Multiply 14·36 by 16·451

16·451

1436

7180

5744

8616

1436

Prod.=236·23636

Ex. 2.

Multiply ·0473 by ·0847

·0847

3311

1893

3784

·00400631=Prod.

The above rule for pointing the decimals in the product may be proved as follows:

$$14·36 = 14\tfrac{36}{100} = \tfrac{1436}{100}$$

$$\text{and } 16·451 = 16\tfrac{451}{1000} = \tfrac{16451}{1000}$$

$$\therefore\ 14·36 \times 16·451 = \tfrac{1436}{100} \times \tfrac{16451}{1000} = \tfrac{23623636}{100000} =$$

$$236\tfrac{23636}{100000} = 236·23636, \text{ as before.}$$

As it only occupies a few seconds to prove any multiplication, it is advisable that all should be proved of which any doubt may be entertained.

The following explains the method of proof.

Ex. 3. Let it be required to multiply 23·33 by 16·321, and prove the work to be right or wrong.

Multiply 23·33 —(A)

by 16·321—(B)

2333

4666

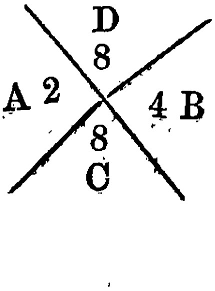
6999

13998

2333

Prod.=380·76893—(C).

PROOF.



Cast the nines out of the sum of the digits in (A), and what is over place on the cross to the left, that is, 2+3+3+3=11=one nine and 2 over, which set down; again, cast the nines out of (B), setting down on the cross to the right what is over, that is, 1+6+3=10=one nine and 1 over, then 1+2+1=4, which does not make up nine, then 4 is over, which set down on the cross at (B); then multiply the numbers on the cross at (A) and (B) together, and cast the nines out of the product, and place what is over on the cross at (D).

Lastly, cast the nines out of the product (C); thus, 3+8=11=9 and 2 over; 2+7=9 and 0 over; 6+8=14=9 and 5 over; 5+3=8, which place on the cross at (C); then, if the numbers on the cross at (C) and (D) are equal, the work is right, and if not, wrong. In casting out the nines in any number, the ciphers or nines need not be taken into account.

It is natural to associate the idea of labour with long detail of execution, but such an idea may be abandoned here, as at most it will not take the operator more than half a minute to prove any multiplication by the directions just given.

EXAMPLES FOR PRACTICE.

Ex. 4. Multiply 3·1416 by 10·24.
Ans. 32·169984.

Ex. 5. Multiply ·00376 by 278.
Ans. 1·04528.

Ex. 6. Multiply 3·1416+·7854 by 2218·192−277·274.
Ans. 7621·984986, true to six places of decimals.

DIVISION OF DECIMALS.

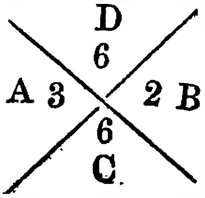
Rule.—Division of decimals is the same as division of whole numbers, only it is to be strictly observed that the number of decimals and ciphers annexed to the dividend must be always equal to the number of decimals both in the divisor and quotient; now, as the number of decimals in the divisor, and also the number of decimals and ciphers affixed to the dividend, are known, the number of decimals in the quotient is determined by making their difference. But as the rule for division of decimals is best drawn from examples, the following are most of the varieties that can occur.

EXAMPLES.

Ex. 1. Divide 3647· by 47.

```
                 Dividend.
     Divisor 47) 3647 (77·595744 Quotient.
                 329
                 ———
                  357
                  329
                  ———
                   280                 210
                   235                 188
                   ———                 ———
                    450                 220
                    423                 188
                    ———                 ———
                     270                 32 Remainder.
                     235
                     ———
                      350
                      329
                      ———
                       210
```

When the divisor, the dividend made less by the remainder, and the quotient are arranged as below, the proof is the same as in multiplication.



```
                     | Ciphers
               Divid | annexed.
               3647  | 000000
Divisor.             |     32 Remainder.          Quotient.
47—(B)         3646·999968—(C)              77·595744—(A)
```

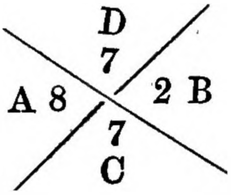
When the divisor is a whole number and less than the dividend, another whole number, the quotient is composed of whole numbers and decimals; the number of decimals in the quotient is equal the number of ciphers called down.

Ex. 2. Divide 47· by 3647.

Divisor. Dividend. Quotient.
3647·) 47·00 (·012887
3647
10530
7294
32360
29176
31840
29176
26640
25529
1111 Remainder.

PROOF.

Quotient = ·012887—(A)
Divisor = 3647·—(B)
Dividend and ciphers affixed.
From 47·000000
Take 1111 Remainder
46·998889—(C)



We obtain five figures in the quotient, but there ought to be six, because there are six ciphers appended to the dividend, and no decimal in the divisor; therefore a cipher must be placed to the left of the figures in the quotient, and then the decimal point.

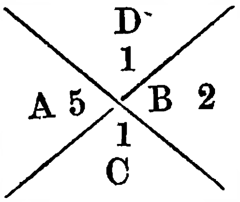
Ex. 3. Divide 347·23 by 5·19878.

Divisor. Dividend. Quotient.
5·19878) 347·2300 (66·79067
3119268
3530320
3119268
4110520
3639146
4713740
4678902
3483800
3119268
3645320
3639146
6174 Remainder.

From	The number of decimals and ciphers appended to the dividend	10
Take	The number of decimals in the divisor . . .	5
Remainder	The number of decimals in the quotient . . .	5

PROOF.

Quotient = 66·79067—(A)
Divisor = 5·19878—(B)
From the dividend and ciphers appended = 347·2300000000
Take the remainder = 6174
347·2299993826—(C)



Ex. 4. Divide 87·3749 by ·00436.

```
·00436 ) 87·3749 ( 20040·1146
         872
         ----
           1749
           1744
           -----
               500
               436
               ----
                 640
                 436
                 ----
                 2040
                 1744
                 -----
                   2960
                   2616
                   ----
                    344 Remainder.
```

From	The number of decimals and ciphers in the dividend	9
Take	The number of decimals in the divisor . . .	5
Remainder	There remains the number of decimals that ought to be in the quotient	4

The proof would be similar to those previously given.

Ex. 5. Divide ·063478 by ·0000125.

·0000125) ·063478 (5078·24
625
978
875
1030
1000
300
250
500
500

Number of decimals and ciphers in the dividend . = 9
Number in the divisor . . . . = 7
The number in the quotient . . . . = 2

☞ To the previous remarks on division we may add the following Rule:—Make the number of decimals in the divisor and dividend equal, by adding ciphers to the deficient one; then, when the divisor divides the dividend without annexing any more ciphers, the quotient will be a whole number; but when we call more ciphers to the dividend, any figures that are put after in the quotient will be decimals.

The reason that the number of decimals in the quotient is equal the number in the dividend with the ciphers appended, made less by the number in the divisor, will appear from the following; taking the 5th example for instance,

$$\text{Dividend} = {\cdot}063478 = \frac{{\cdot}063478}{1000000} = \frac{63478}{1000000}$$

$$\text{Divisor} = {\cdot}0000125 = \frac{{\cdot}0000125}{10000000} = \frac{125}{10000000}$$

$$\therefore \quad {\cdot}063478 \div {\cdot}0000125 = \frac{63478}{1000000} \div \frac{125}{10000000} =$$

$$\frac{63478}{1000000} \times \frac{10000000}{125} = \frac{634780000000}{125000000} = \frac{634780}{125} =$$

$$5078\frac{30}{125} = 5078\frac{6}{25} = 5078\frac{24}{100} = 5078{\cdot}24.$$

EXAMPLES FOR PRACTICE.

Ex. 6. Divide 354· by 3·1416. *Ans.* 112·681.

Ex. 7. Divide 3·1416 by 89·74. *Ans.* ·0350078.

Ex. 8. Divide 2218·192—·7854 by 277·274+3·1416× ·5236. *Ans.* 15·10236.

Those who have studied the history of the mathematical sciences cannot but have noticed the slow manner in which improvements have been admitted into general use; even at this late date, barbarous and inadequate as the method was, the author was obliged to allow his bases to be expressed in vulgar fractions, in order that his published works on Dual Arithmetic might be better understood. Dual Arithmetic calls for but few innovations to establish its general notation; decimal arithmetic is left in possession of the full point (·), while in dual arithmetic the comma (,) is employed, a distinction easily remembered. The first notice of decimals is to be found in a tract at the end of Stevinus's *Arithmétique*, in the collection of his works by his friend and pupil Albert Girard; the tract is entitled *La Disme.* This collection was first published in Flemish, about the year 1590. At this early date, decimals in the first place are termed primes, and marked (1); those in the second place are marked (2), and called seconds, and so on; whilst all integers are characterised by the sign (0), which is put after or above the last digit.

EXAMPLE IN ADDITION.

1590.	1865.
(0)(1)(2)(3)(4)	
3 4 6 1 2	3·4612
21 4 7 7 2	21·3472
1 3 0 0 6	1·3006
24 0 0 4 9	24·0049
50 1 1 3 9	50·1139
(0) (1) (2) (3) (4)	

The denominators 10, 100, 1000, &c., were employed after the time of Briggs and Napier.

DUAL ARITHMETIC.

Dual arithmetic is *a new art of manœuvring numbers* and investigating the relations of quantities with ease and accuracy, with or without the use of tables.

The term *dual* is employed because the art has *two branches*, the basis of each branch being composed of *two parts*, and because the digits of a dual number may be subjected to a variety of changes in magnitude and position, while at the same time remaining equal in value to *two unchangeable extremes*, namely, a natural number and its logarithm to a known base.

Bases of the Ascending Branch.

The bases of this branch of the art can be expressed as follows:

$$+\infty\,.\,.\,.\,.\,.\,(10000+1);\ (1000+1);\ (100+1);\ (10+1);\ (1+1);\ (\tfrac{1}{10}+1);\ (\tfrac{1}{100}+1);\ (\tfrac{1}{1000}+1);\ .\,.\,.\,.\,.\,.\,1;$$

more conveniently written thus:

$$+\infty\,.\,.\,.\,.\,10001;\ 1001;\ 101;\ 11;\ 2;\ 1{\cdot}1;\ 1{\cdot}01;\ 1{\cdot}001;\ .\,.\,.\,.\ 1 \text{ the limit,}$$

increasing in magnitude from right to left.

These bases are less and less as they approach 1, but cannot be less than 1.

Bases of the Descending Branch.

These bases may be expressed thus:

$$-\infty\,.\,.\,.\,.\,(1-1000);\ (1-100);\ (1-10);\ (1-1;\ (1-\tfrac{1}{10});\ (1-\tfrac{1}{100});\ (1-\tfrac{1}{1000});\ .\,.\,.\,(1),$$

but more concisely written,

$$-\infty\,.\,.\,.\,.\,-999;\ -99;\ -9;\ 0;\ {\cdot}9;\ {\cdot}99;\ {\cdot}999;\ .\,.\,.\,.\,.\,.\,.\,1,$$

decreasing in magnitude from right to left.

This scale of bases approaches 1, but cannot be greater than 1.

An example will make clear anything that may seem too abstract in the preceding definitions.

The diameter of the earth through the poles is said to be 41706091·152 ft., which is a conventional form of expressing

$$4 \text{ times } (10000000\cdot) + \text{once } (1000000\cdot) + 7 \text{ times } (10000\cdot) + 6 \text{ times } (1000\cdot) + \text{nine times } (10\cdot) + 1\cdot + \tfrac{1}{10} + \tfrac{5}{100} + \tfrac{2}{1000},$$

by the notation of common arithmetic, which, according to the ordinary method agreed upon to express powers, becomes

$$4(10)^7 + (10)^6 + 7(10)^5 + 6(10)^3 + 9(10)^1 + 1 + (10)^{-1} + 5(10)^{-2} + 2(10)^{-3}.$$

In this notation of common arithmetic 4, 1, 7, &c., are termed digits, and do not exceed 9.

In dual arithmetic the powers of the dual bases are only registered. Thus 41706091·152 is equal to

$$(\cdot 99999)^1 (\cdot 999999)^3 (\cdot 9999999)^3 (\cdot 99999999)^6 (1+1)^2 (1\cdot 01)^4 (1\cdot 001)^2, \text{ when multiplied by } 10^7.$$

The bases being omitted, this dual number is written

'0'0'0'0'1'3'3'6 ↑ $10^7 2^2$ ↓ 0,4,2,0,0,0,0,0, ; (A).

The natural number 41706091·152 is also equal to the dual number

$(10)^7(2)^2$ ↓ 0,4,1,9,8,6,9,6, ;

it is likewise equal to the dual number

'8'3'1'4'6'8'1'0 ↑ $(10)^8$; and

to an immense number of other dual numbers.

Referring to the form (A), '1'3'3'6; 0,4,2, &c., are called dual digits, and express the powers of the bases involved, and, unlike the digits of ordinary arithmetic, may be greater or less than 9. For example, the common number 41706091·152 is equal to the dual number

'0'0'1'5'7'0'8'4 ↑ 10^7 ↓ 15,0,0,0,0,0,0,0,

The zero between ↓ and 4, in (A), shows that no power of 1·1 is employed; the same remark applies to zero and other bases.

A dual number of positive dual digits has always an exact value in common numbers when no contractions are employed in the reduction.

When eight positions to the right and eight to the left of the signs ↑ ↓, counting from left to right in both cases, are occupied by ciphers or other digits, the sign ↓ being placed before the eight ascending digits and ↑ after the eight descending, yet, with respect to range, the dual number is said to be one of eight digits, although sixteen positions, and other positions between the signs ↑ and ↓, may be occupied.

If one of the signs ↑ or ↓ is omitted, the positions attached to the other are supposed to be occupied by ciphers.

☞ When the last dual digit and all that follow are rejected, and when the last digit is 5, 6, 7, 8, or 9, the digit preceding may be counted one more, as in decimal arithmetic.

For most practical purposes common arithmetical results are required true, to not more than seven places of figures. To obtain this degree of accuracy, eight consecutive dual digits must be employed. In making calculations the allowances specified must be attended to.

A dual number is easily transformed into another, all of whose digits being reduced to ciphers, except the last. The transformation of a dual number of eight digits into another, whose first seven digits are ciphers, is termed *reducing a dual number to the eight position.* A dual number reduced to the eight position is called a dual logarithm.

$$2{\cdot} = \downarrow 7,2,6,0,7,8,2,6, = \downarrow 0,0,0,0,0,0,0,69314718,$$
$$= \downarrow {}^{8}69314718.$$

In practice the 8 is omitted, and the expression is written

$$2{\cdot} = \downarrow 69314718, \text{ which represents } (1{\cdot}00000001)^{69314718}.$$

Then 69314718 is termed the dual logarithm of 2·, written

$$\downarrow,\ (2{\cdot}) = 69314718.$$

Then the dual logarithm of 41706091·152 is equal to the whole number 1615789463, or

$$\downarrow,\ (41706091{\cdot}152) = 1615789463,.$$

The dual logarithms of common numbers are easily found, as well as the common numbers corresponding to dual logarithms, without the use of tables.

NOTATION. ASCENDING BRANCH.

The notation, although new, is easily remembered, from its symmetry, compactness, and uniformity.

1 is represented by $\downarrow$.

ONE DECIMAL. **FIRST POSITION.**

1·1 is represented by $\downarrow 1$,
$(1{\cdot}1)^2$ ” ” $\downarrow 2$,
$(1{\cdot}1)^3$ ” ” $\downarrow 3$,
&c. &c.

TWO DECIMALS. **SECOND POSITION.**

(1·01) is represented by $\downarrow 0, 1$, or $\downarrow {}^{2}1$,
$(1{\cdot}01)^2$ ” ” $\downarrow 0, 2$, or $\downarrow {}^{2}2$,
$(1{\cdot}01)^3$ ” ” $\downarrow 0, 3$, or $\downarrow {}^{3}3$,
&c. &c.

THREE DECIMALS. **THIRD POSITION.**

(1·001) is represented by $\downarrow 0, 0, 1$, or $\downarrow {}^{3}1$,
$(1{\cdot}001)^2$ ” ” $\downarrow 0, 0, 2$, or $\downarrow {}^{3}2$,
$(1{\cdot}002)^3$ ” ” $\downarrow 0, 0, 3$, or $\downarrow {}^{3}3$,
&c. &c.

$(1{\cdot}1)^5(1{\cdot}01)$ is represented by $\downarrow 5, 1$,
$(1{\cdot}1)^7(1{\cdot}01)^2$ ” ” $\downarrow 7, 2$,
$(1{\cdot}1)^3(1{\cdot}01)^4(1{\cdot}001)^5$ ” ” $\downarrow 3, 4, 5$,
$(1{\cdot}001)^6(1{\cdot}00001)^2(1{\cdot}00000001)^8$ is expressed by
$\downarrow 0,0,6,0,2,0,0,8$,

$\downarrow$ 0,0, in the first and second positions, indicates that no power of (1·1) or of (1·01) is involved. The cipher in the fourth position indicates that no power of 1·0001 is involved; the same may be said of the other positions.

NOTATION. DESCENDING BRANCH.

In this branch the arrow points *up*, and the comma is to the left of the digit and *above*, while in the ascending branch the arrow points *down*, and the comma is to the right of the digit and *below*.

1 is represented by $\uparrow$

One decimal (·9) ” ” ’1 $\uparrow$ in the first position.
” ” $(\cdot 9)^2$ ” ” ’2 $\uparrow$ ” ” ”
” ” $(\cdot 9)^3$ ” ” ’3 $\uparrow$ ” ” ”
&c. &c.

TWO DECIMALS. SECOND POSITION.

$(\cdot 99)$ is represented by $'0'1\uparrow$ or $'1_2\uparrow$
$(\cdot 99)^2$ " " $'0'2\uparrow$ or $'2_2\uparrow$
$(\cdot 99)^3$ " " $'0'3\uparrow$ or $'3_2\uparrow$
$(\cdot 99)^4$ " " $'0'4\uparrow$ or $'4_2\uparrow$
&c. &c.

THREE DECIMALS. THIRD POSITION.

$(\cdot 999)$ is represented by $'0'0'1\uparrow$ or $'1_3\uparrow$
$(\cdot 999)^2$ " " $'0'0'2\uparrow$ or $'2_3\uparrow$
$(\cdot 999)^3$ " " $'0'0'3\uparrow$ or $'3_3\uparrow$
&c. &c.

☞ In both branches, if there be *n* decimals in any base, its powers or dual digits are placed in the *n*th position. ☜

$(9)^3(\cdot 99)^2$ is written $'3'2\uparrow$
$(9)^7(\cdot 99)^5$ " $'7'5\uparrow$
$(\cdot 999)^3(\cdot 999999)^2(\cdot 99999999)^6$ is written $'0'0'3'0'0'2'0'6\uparrow$

A cipher being in the first and also in the second position shows that no power of ·9 or ·99 is employed; the same may be said of other positions occupied by ciphers.

$'0'0'3'0'0'2'0'6'\uparrow$ may be written $'3'0'0'2'0'6_3\uparrow =$
$'3_3\uparrow\,'2_6\uparrow\,'6_8\uparrow$

In the author's works, "Dual Arithmetic: a New Art," Parts I. and II., and in the "Young Dual Arithmetician," without the use of tables, in a variety of ways, and under different circumstances, it is shown, by easy, independent, and direct processes, how any two of the three corresponding numbers

(NATURAL NUMBER); (DUAL NUMBER); (DUAL LOGARITHM);

might be found, the remaining one being given.

Any one of these convertible numbers being given, the other two may be also found by employing Tables I. and II. (see "Young Dual Arithmetician"). Table I. is of the ascending branch, in which the natural numbers range from

1· to 2·99161136.

Table II. is of the descending branch, of which the natural numbers range from

·29916114 to 1.

Hence, when these tables are employed, the powers of the base 1 + 1 or 2 are dispensed with, and only the powers of 10 retained. When operations are performed with dual numbers in their lowest terms, and tables used, it is not necessary that such tables should range beyond the natural numbers

from 1·41421356 to 1, and from 1 to ·70710678.

(See "Tables of Dual Logarithms and of Angular Magnitudes and Trigonometrical Lines.")

CONSTRUCTION OF TABLES OF DUAL LOGARITHMS.

In constructing Tables of dual logarithms, no arithmetical operations beyond those of addition and subtraction are necessary.

To find the natural numbers corresponding to any dual number of one digit.

Adding, Ascending branch.	Subtracting, Descending branch.
↓ 1, = 1·1	'1 ↑ = ·9
11	9
↓ 2, = 1·21	'2 ↑ = ·81
121	81
↓ 3, = 1·331	'3 ↑ = ·729
1331	729
↓ 4, = 1·4641	'4 ↑ = ·6561
14641	6461
↓ 5, = 1·61051	'5 ↑ = ·59049
161051	59049
↓ 6, = 1·771561	'6 ↑ = ·531441
1771561	531441
↓ 7, = 1·9487171	'7 ↑ = ·4782969
19487171	4782969
↓ 8, = 2·14358881	'8 ↑ = ·43046721
214358881	&c.
↓ 9, = 2·357947691	
&c.	

The above tables are formed by simple addition and subtraction, each line being obtained from the preceding one by commencing a figure further to the right.

To find the natural numbers corresponding to dual numbers of two digits.

Ascending, adding.	Descending, subtracting.
↓ 0,1 = 1·0 1	’0’1 ↑ = ·99
10 1	99
↓ 0,2 = 1·0 20 1	’0’2 ↑ = ·98 01
10 20 1	98 01
↓ 0,3 = 1·0 30 30 1	’0’3 ↑ = ·97 02 99
10 30 30 1	97 02 99
↓ 0,4 = 1·0 40 60 40 1	’0’4 ↑ = ·96 05 96 01
10 40 60 4	96 05 96 01
↓ 0,5 = 1·0 51 01 00 5	’0’5 ↑ = ·95 09 90 04 99
&c. = &c.	&c.

Here commencing with ↓ 0, 1, = 1·01 and ’0’1 ↑ = ·99, we pursue the same process as before, only placing the figures to be added to or subtracted from the preceding lines two figures further to the right, instead of one, neglecting all figures after the eight decimal place.

When the first dual digit is greater than one, we start with the value of that digit. Thus:

Ascending branch.	Descending branch.
↓ 6, = 1·7 71 56 1	’6 ↑ = ·53 14 41
17 71 56 1	53 14 41
↓ 6,1, = 1·7 89 27 66 1	’6’1 ↑ = ·52 61 26 59
17 89 27 7	52 61 27
↓ 6,2, = 1·8 07 16 93 8	’6’2 ↑ = ·52 08 65 32
18 07 16 9	52 08 65
↓ 6,3, = 1·8 25 24 10 7	’6’3 ↑ = ·51 56 56 67
18 25 24 1	51 56 57
↓ 6,4 = 1·8 43 49 34 8	’6’4 ↑ = ·51 05 00 10
18 43 49 3	51 05 00
↓ 6,5, = 1·8 61 92 84 1	’6’5 ↑ = ·50 53 95 10
&c. = &c.	&c. = &c.

In the same way the common numbers corresponding to all dual numbers of two digits may be computed and tabulated.

Ascending.	Descending.
Again, ↓ 0,0,1 = 1·00 | 1	'0'0'1 ↑ = ·999 |
| 100 | 1	| 999
↓ 0,0,2 = 1·00 | 200 | 1	'0'0'2 ↑ = ·998 | 001 |
| 100 | 200	| 998 | 00
↓ 0,0,3 = 1·00 | 300 | 300	'0'0'3 ↑ = ·997 | 003 | 00
| 100 | 300	| 997 | 00
↓ 0,0,4 = 1·00 | 400 | 600	'0'0'4 ↑ = ·996 | 096 | 00
| 100 | 400	| 996 | 01
↓ 0,0,5 = 1·00 | 501 | 000	'0'0'5 ↑ = ·995 | 009 | 99
| 100 | 501	| 995 | 01
↓ 0,0,6 = 1·00 601 501	'0'0'6 ↑ = ·994 014 98
&c. = &c.	&c. = &c.

Operating the same as before, only setting the line to be added or subtracted three figures back.

Ascending.	Descending.
↓ 6,5,0 = 1·86 | 192 | 841	'6'5'0 = ↑ ·505 | 395 | 10
| 186 | 193	| 505 | 40
↓ 6,5,1 = 1·86 | 379 | 034	'6'5'1 = ↑ ·504 889 | 70
| 186 | 379	| 504 | 89
↓ 6,5,2 = 1·86 | 565 | 413	'6'5'2 = ↑ ·504 384 81
| 186 | 565	504 38
↓ 6,5,3 = 1·86 751 978	'6'5'3 = ↑ ·503 880 43
&c. = &c.	&c. = &c.

In this manner the common numbers of all dual numbers of three digits can be found and tabulated.

Ascending.	Descending.
↓ 0,0,0,1 = 1·000 | 1	'0'0'0'1 ↑ = ·9999 | 0000
| 1000 | 1	| 9999
↓ 0,0,0,2 = 1·000 | 2000 | 1	'0'0'0'2 ↑ = ·9998 | 0001
| 1000 | 2	| 9998
↓ 0,0,0,3 = 1·000 3000 3	'0'0'0'3 ↑ = ·9997 0003
&c. = &c.	&c. = &c.

Or the natural numbers corresponding to dual numbers of four digits are found as those of three digits, setting the line to be added or subtracted four figures back.

Ascending.				Descending.		
↓ 6,5,2,0	=1·865	6541 1865	3 7	’6’5’2’0 ↑	=·5043	8481 5044
↓ 6,5,2,1	=1·865	8407 1865	0 8	’6’5’2’1 ↑	=·5043	3437 5043
↓ 6,5,2,2	=1·866	0272 1866	8 0	’6’5’2’2 ↑	=·5042	8394 5043
↓ 6,5,2,3	=1·866	2138 1866	8 2	’6’5’2’3 ↑	=·5042	3351 5042
↓ 6,5,2,4	=1·866	4005 1866	0 4	’6’5’2’4 ↑	=·5041	8309 5042
↓ 6,5,2,5	= 1·866	5871	4	’6’5’2’5 ↑	=·5041	3267
&c.	=&c.			&c.	=&c.	

In this way, by common addition and subtraction alone, the natural numbers corresponding to the dual numbers can be obtained throughout tables of both ascending and descending dual numbers.

NEXT TO COMPUTE THE CORRESPONDING DUAL LOGARITHMS.

Ascending.

The dual logarithm of ↓ 1, =9531018,
" " ↓ 0,1, = 995033,
" " ↓ 0,0,1, = 99950,
" " ↓ 0,0,0,1, = 10000,

Descending.

’10536052=dual log of ’1 ↑
’1005034= " " ’0’1 ↑
’100050= " " ’0’0’1 ↑
’10001= " " ’0’0’0’1 ↑

For the calculation of these numbers, see "Dual Arithmetic," Part II., and the "Young Dual Arithmetician." From the above numbers all the dual logarithms of both branches can be found by simple addition.

Ascending.	Descending.
log ↓ 1,= 9531018	’10536052=log of ’1 ↑
9531018	10536052
log ↓ 2,=19062036	’21072104= „ ’2 ↑
9531018	10536052
log ↓ 3,=28593054	’31608156= „ ’3 ↑
9531018	10536052
log ↓ 4,=38124072	’42144208= „ ’4 ↑
9531018	10536052
log ↓ 5,=47655090	’52680260= „ ’5 ↑
9531018	10536052
log ↓ 6,=57186108	’63216312= „ ’6 ↑
&c.=&c.	&c.=&c.

Or the dual logarithms of all dual numbers of one digit in the first position may be found by the successive addition of the number 9531018, for the ascending branch, and ’10536052 for the descending.

Ascending.	Descending.
log ↓ 0,1,= 995033	’1005034=log of ’0’1 ↑
995033	1005034
log ↓ 0,2,= 1990066	’2010068= „ ’0’2 ↑
995033	1005034
log ↓ 0,3,= 2985099	’3015102= „ ’0’3 ↑
995033	1005034
log ↓ 0,4,= 3980132	’4020136= „ ’0’4 ↑
&c.=&c.	&c.=&c.
and log ↓ 6, =57186108	’63216312=log of ’6’0 ↑
995033	1005034
log ↓ 6,1,=58181141	’64221346= „ ’6’1 ↑
995033	1005034
log ↓ 6,2,=59176174	’65226380= „ ’6’2 ↑
995033	1005034
log ↓ 6,3,=60171207	’66231414= „ ’6’3 ↑
995033	1005034
log ↓ 6,4,=61166240	’67236448= „ ’6’4 ↑
&c.=&c.	&c.=&c.

Thus the dual logarithms of dual numbers of two digits are derived from those of one digit by the successive addition of the number 995033, for the ascending, and ’1005034 for the descending branch.

Ascending.	Descending.
log ↓ 0,0,1,= 99950,	’100050=log of ’0’0’1 ↑
99950,	100050
log ↓ 0,0,2,= 199900,	’200100= „ ’0’0’2 ↑
99950,	100050
log ↓ 0,0,3,= 299850,	’300150= „ ’0’0’3 ↑
99950,	100050
log ↓ 0,0,4,= 399800,	’400200= „ ’0’0’4 ↑
&c. = &c.	&c. = &c.
but log ↓ 6,4,0,=61166240,	’67236448=log of ’6’4’0 ↑
99950,	100050
log ↓ 6,4,1,=61266190,	’67336498= „ ’6’4’1 ↑
99950,	100050
log ↓ 6,4,2,=61366140,	’67436548= „ ’6’4’2 ↑
99950,	100050
log ↓ 6,4,3,=61466090,	’67536598= „ ’6’4’3 ↑
&c. = &c.	&c. = &c.

Thus the dual logarithms of all dual numbers of three digits are derived from those of two digits by the successive addition of the numbers 99950, and ’100050.

Ascending.	Descending.
log ↓ 0,0,0,1,= 10000,	’10001=log of ’0’0’0’1 ↑
10000,	10001
log ↓ 0,0,0,2,= 20000,	’20002= „ ’0’0’0’2 ↑
10000,	10001
log ↓ 0,0,0,3,= 30000,	’30003= „ ’0’0’0’3 ↑
10000,	10001
log ↓ 0,0,0,4,= 40000,	’4004= „ ’0’0’0’4 ↑
&c. = &c.	&c. = &c.
but log ↓ 6,4,3,0,=61466090,	’67536598= „ ’6’4’3’0 ↑
10000,	10001
log ↓ 6,4,3,1,=61476090,	’67546599= „ ’6’4’3’1 ↑
10000,	10001
log ↓ 6,4,3,2,=61486090,	’67556600= „ ’6’4’3’2 ↑
10000,	10001
log ↓ 6,4,3,3,=61496090,	’67566601= „ ’6’4’3’3 ↑
&c. = &c.	&c. = &c.

Or the dual logarithms of all dual numbers of four digits may be derived from those of three digits by the successive addition of the numbers 10000, and '10001.

With the exception, therefore, of the calculation of the dual logarithms of ↓1,; ↓0,1; ↓0,0,1,; ↓0,0,0,1,; and of '1↑; '0'1↑; '0'0'1↑; '0'0'0'1↑, and of the dual logarithms of 10 and 2—all of which can be verified by methods shown in the author's works on the art and science of dual arithmetic in ten minutes—every result in the tables of dual numbers, their corresponding natural numbers, and dual logarithms, can be calculated, as here shown, by simple addition and subtraction, without the aid of any other arithmetical operation.

Having said enough to induce a student to examine dual arithmetic; in conclusion we propose to arouse the attention of accomplished mathematicians, in solving by a direct process the following question. In order that this solution may not be passed over lightly, we have selected a question that could not be solved by all the living mathematicians, and all who ever lived, with the help of their numerous theorems, without the aid of dual arithmetic.

Question.

Find a number which, when added to the circumference of a circle whose diameter = 10000, the sum and its common logarithm, each, is composed of the same consecutive digits?

$$\pi = 3{\cdot}14159265\ldots\ldots$$

Put $a=10000$, and $m=10^4$, and let x be the required number.

$$\text{Then } 10^{\frac{a\pi+x}{10^4}} = a\pi+x$$

$$\therefore \left(10^{\frac{1}{m}}\right)^{a\pi+x} = a\pi+x.$$

$$\therefore \frac{1}{\left(10^{\frac{1}{m}}\right)^{a\pi+x}} = \frac{1}{a\pi+x}.$$

taking $a\,\pi+x$ root of both sides of this equation, we have

$$\frac{1}{10^{\frac{1}{m}}} = \left(\frac{1}{a\,\pi+x}\right)^{\frac{1}{a\,\pi+x}}.$$

Putting b for $\frac{1}{10^{\frac{1}{m}}}$ and z for $\frac{1}{a\,\pi+\pi}$, the last equation becomes $z^z=b$, the general solution of which is given in "Dual Arithmetic," Part II., pp. 98-100. We find

$$x=15276{\cdot}54179287965.$$

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